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On some musical applications of Ircam's "Mathematical School for Musicians and other Non-Mathematicians"

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In this paper we provide a brief overview of the two mathematical constructions presented by Yves André during the first two session of Ircam's "Mathematical School for Musicians and other Non-Mathematicians". After discussing a musical application of Operator Theory we try to suggest the possible connection between Jean-Yves Girard's *Geometry of Interaction* (based on a special case of von Neumann algebras) and a recent topos-theoretical model of musical gesture [28] which provides an extension of the "classical" structure-oriented framework of Guerino Mazzola's *Topos of Music*. This interpretation raises the question of the articulation between formal models, programming and computation in music and suggests some possible directions in the definition of "musical logic".

Introduction

The current sixth season (2006-2007) of Ircam's MaMuX Seminar (Mathematics/Music and relations to other fields) hosts a special "Mathematical School" addressed to musicians and non-mathematicians.[1] The aim of the School is at describing and making accessible some central concepts of contemporary mathematics to a non-professional audience. These concepts are presented by the mathematician Yves André (ENS/CNRS) who discusses their relevance to the mathematical activity without neglecting neither the specificity of the mathematical writing nor its demonstrative component. We focus on the two mathematical constructions presented by Yves André during the first two session of the school: von Neumann algebras and Grothendieck's Topos Theory.[2] After briefly discussing the place of von Neumann algebras with respect to other mathematical disciplines (section 1), we describe one recent musical applications of spectral theory concerning an open problem in mathematics whose musical metamorphosis is an active research field in our "mathemusical" community (section 2). Section 3 provides an overview of Grothendieck Topos Theory, whereas a recent musical example is discussed in section 4 by briefly presenting the new categorical model of musical gestures proposed in [28]. We end by suggesting some possible connections between von Neumann algebras and this topos-theoretical construction.

1 Overview on the Operator Theory: an introduction to von Neumann algebras

By following Yves André's first presentation [3], an original way of introducing von Neumann algebras consists of considering this topic as the crossing point between two fields: linear algebra (via the passage

[1]This School is organized in conjunction between the MaMuPhi Seminar (Mathematics, Music and Philosophy) of the École Normale Supérieure of Paris (dir. François Nicolas, Charles Alunni and Moreno Andreatta) and the MaMuX Seminar that we organize at Ircam since 2001.

[2]In the oral presentation during the "Mathematics and Computation in Music" Conference, we will also discuss the remaining topic of the school: Galois theory.

to infinit dimension) and the mesure theory (via the passage to non-commutativity). This crossing point will be the initial object of an arrow leading naturally to non-commutative geometry (see Figure 1).

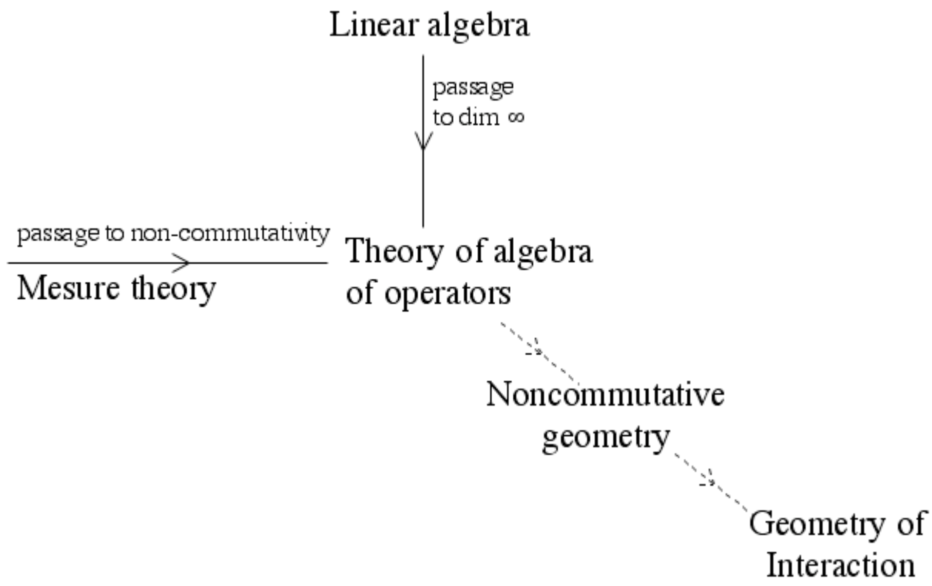

Figure 1. The place of the theory of operators with respect to linear algebra, mesure theory and non-commutative geometry.

2 On a musical application of Operator Theory: the spectral conjecture and its "canonical" metamorphoses

Although Fuglede's researches in Operator Theory lead to interesting solution of some open problems in von Neumann algebras [14], it turned out that one of the question he raised in the Seventies still remains an open conjecture in mathematics. Fuglede's Conjecture (also called the spectral conjecture) concerns the relation between *spectral* sets and *tiling* properties. Fuglede conjectured that a finite Lebesgue measurable subset A of $\mathbb{R}^n$ is spectral (i.e. the family $\{e^{2\pi i x\lambda}\}_{\lambda\in\Lambda\subset\mathbb{R}^n}$ is an orthogonal basis of the Hilbert space $L^2(A)$ of square-integrable measurable functions on A) iff A tiles $\mathbb{R}^n$ by translation. Although Fuglede Conjecture has been proved in a great number of special cases, it still remain open for $n = 1$. This is the precisely the interesting case for the rhythmic canons construction. By using Coven-Meyerowitz conditions [12], Emmanuel Amiot has recently shown that it is sufficient to prove these two conditions for Vuza's Regular Complementary Canons of Maximal Category (shortly RCMC-Canons) i.e. the tiling canons that can be formalized as the factorization of a cyclic group into two non-periodic subsets.[1] This result provides a general framework where one can analyse the deep connections between the musical metamorphoses of Fuglede spectral conjecture and a number-theoretical conjecture raised by Minkowski at the end of XIX$^{\mathtt{th}}$ Century (see the diagram in Figure 2).[2]

Despite the "mathemusical" interest of this open problem, this particular application to Operator Theory to music does not change the notion of a (musical) point in a (musical) space. RCMC-Canons are periodic *global* structures[3] obtained by temporal translation of a tiling set. The problem ramains open to consider musical compositional processes leading to tiling structures that are, for example, quasi-periodic. One of the most intriguing example is the Penrose tiling that can be formalized within Connes' noncommutative geometry. In fact the density of the two different tiles in a Penrose tiling can be measured by using the

[1]See [1] for the connection between Vuza's canon and Fuglede Conjecture.

[2]See [5] and [6] for a (relatively)comprehensive presentation of the musical metamorphosis of Minkowski's number-theoretical and geometric Conjectures via Hajos algebraic solution and Vuza's model of RCMC-Canons.

[3]In the sense of Mazzola's "global compositions" [25].

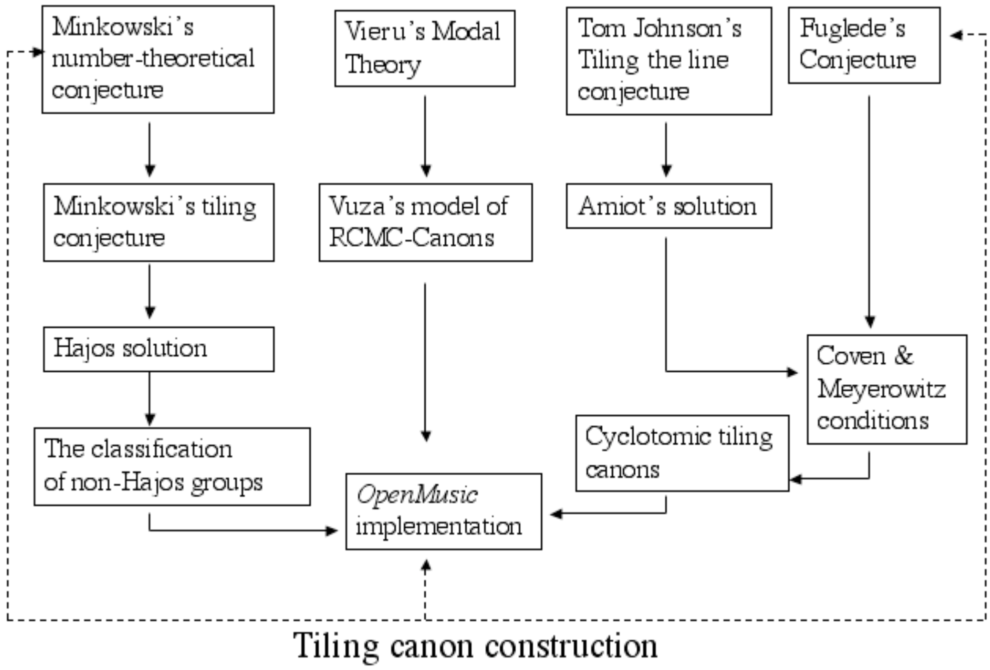

Figure 2. A diagram summarizing the deep connections between Minkowski's and Fuglede's conjecture, via the tiling canon construction.

noncommutative C^*-algebra associated to the quotient space $\mathcal{K}/\mathcal{R}$ of a compact space $\mathcal{K}$ (homeomorphic to the Cantor set) and a given equivalence relation $\mathcal{R}$.

3 Overview on Grothendieck Topos Theory

As shown by Yves André in his second presentation [4], one can follow the idea described in section 1 and introduce Grothendieck Topos Theory as (the prolongation of) the crossing point of two disciplines: general topology (via the "categorification") and Riemann surfaces (via the "sheafication"). Moreover, as much as Alain Connes' noncommutative geometry provides the basis of a new paradigm in logic (geometry of interaction), Grothendieck topos naturally merges into a new logical architecture: intuitionnistic logic (see Figure 3). In both cases, noncommutative geometry as well as Grothendieck topos lead to a redefinition of the concept of *point* in a space. In fact, Grothendieck redefines a point as being a morphism $f : X \to Y$ in the category of schemes, i.e. topological spaces together with commutative rings for all their open sets, arising from "gluing together" spaces of prime ideals (or *spectra*) of commutative rings. Conversely a morphism $f : X \to Y$ in any category can be viewed as a point in the presheaf associated with Y. This is one of the multiple applications of the Yoneda lemma. This lemma, which can be seen as a broad generalization of Cayley's theorem[1] from group theory, concerns the isomorphism between the family $Nat(\mathcal{C}(-, X), F)$ of natural transformations from $\mathcal{C}(-, X)$ to F where F is a (contravariant) functor from a given (locally small) category C to the category Set of sets and X is an object of C.

4 On a *dynamical* musical application of topos theory: musical gestures as processes

As in section 2, we focus on one recent application of Grothendieck's topos theory to music. This construction provides a radical shift in mathematical music theory from the topos-theoretical formalization of

[1]Cayley's theorem assures that a finite group G is isomorphic to a subgroup of the symmetric group on G, i.e. the set of all bijective functions from G to G.

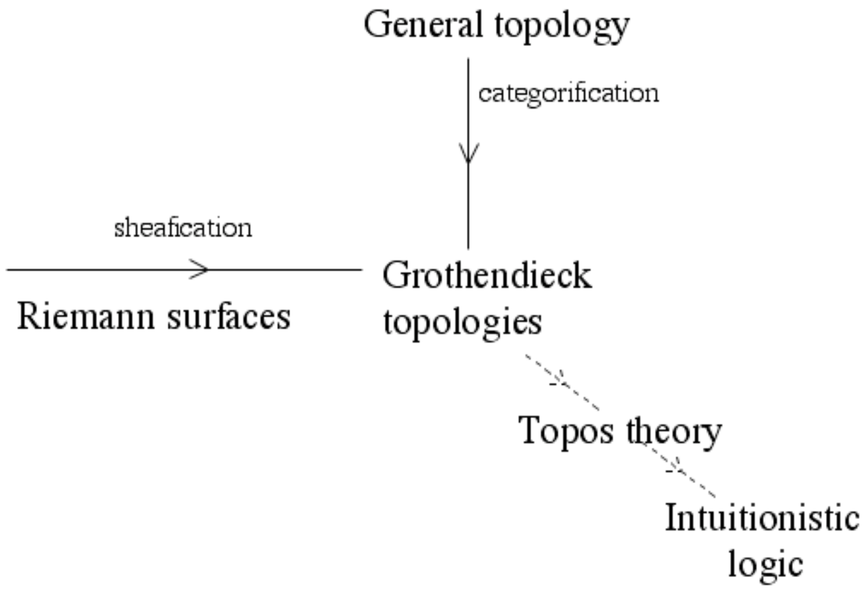

Figure 3. The place of topos theory with respect to topology and Riemannian geometry.

musical *structure* (as summarized by [25]) to a mathematical theory of musical *processes*.[2] This demands to abandon the "static" structure of modules over a given ring to concentrate instead to the category of *directed graphs* of curves in topological spaces. It is interesting to notice that this category has also been recently proposed as a foundational concept in mathematics for both classical and categorical set-theory [13]. Within this new category, Grothendieck's concept of point, that we sketched in the previous section, suggests that a morphism can be viewed as the movement of an arrow. In other words, as the authors of [28] emphasized, *the gesture is a morphism, where the linkage is a real movement and not only a symbolic arrow without bridging substance*. By means of the concept of adjointness between functors, it is possible to musically link the category of directed graphs to the category of spectroids (mathematical formulae) and the category of diagrams of curves in topological space (category of gestures). The adjoint relation can be expressed by saying that by means of diagrams (i.e. systems of transformational arrows), mathematics turns gestures into formulas and conversely, musical activity "unfreezes" formulas into gestures.[3]

5 Towards a geometry of interaction for musical processes

There are many connections between the two main topic presented in this article. As suggested by Pierre Cartier [9] and Alain Connes [11] noncommutative geometry and topos theory are linked together by the concept of *grupoid* which is a generalization of the group structure. In order to discuss the musical application of the topics that we presented in this paper, we start by extending their diagram by also including David Lewin's *Generalized Interval System* construction (based on the group action) and K-nets (whose categorical interpretation can be taken as a starting point for the new process-oriented approach in mathematical music theory). As we said, it is possible to further extend the diagram by also including Jean-Yves Girard's geometry of interaction as one of the possible logical interpretation of Operator Theory. We conjecture that this extension might provide a new point of view for a definition of a "musical logic" (see Figure 4).

Although the relation between music and logic has been addressed in many writings, most of them focuse

[2]The relevance of topos theory for music analysis has also been discussed by Thomas Noll by focusing on the topos of monoid actions [30]. An intermediate step between the structure-oriented and the process-oriented application of topos theory to music is provided by the interpretation of K-nets as limit denotators [27].

[3]See [28] for a formalized presentation of this metaphorical idea.

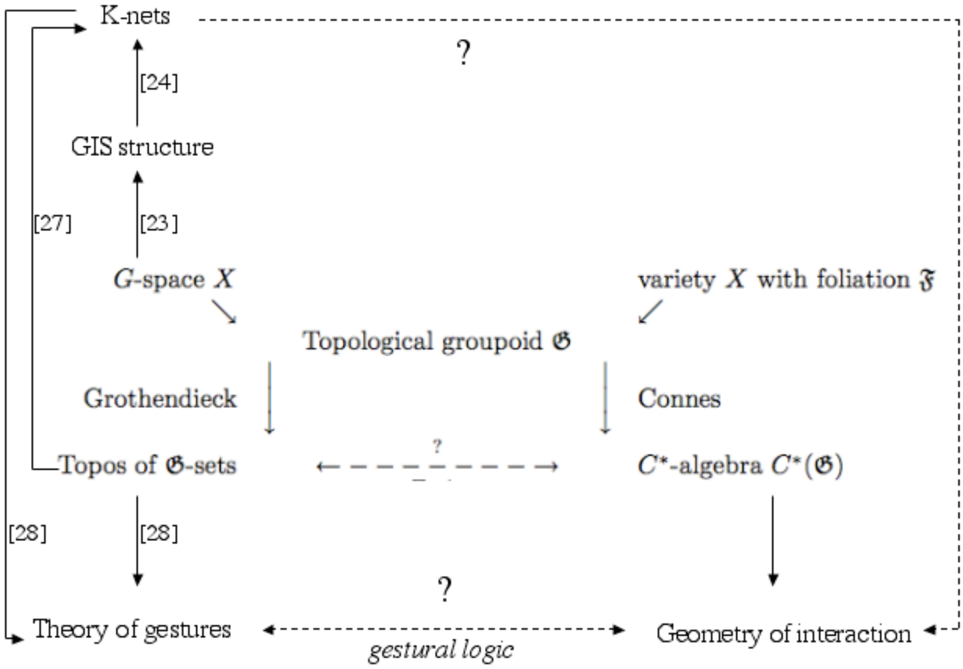

Figure 4. A diagram summarizing the possible connections between a theory of musical gestures (as a ramification of topos theory) and the geometry of interaction (as a logical interpretation of noncommutative geometry).

almost exclusively on the logical formalization of music theory.[1] We would like to stress the fact that the question of a possible definition of the "musical logic" is much more general that the problem of the mathematical (or logical) foundation of music theory.[2] This question has been addressed by the authors of [28] by introducing the concept of *gestural logic* as a special case of Heyting logic. This might provide not only a logical foundation for the relations between transformational music theory and gesture theory but also a possible musical application of Jean-Yves Girard's geometry of interaction. This "new form of semantics" [17] introduces a *procedural* level within the logical operations by radically generalizing the intuitionistic view of proof as *functions* to the paradigm of proof as *actions*.[3] By claiming that "operator algebra is more primitive than set theory" [19], the geometry of interaction provides a mathematical foundation of *truth* that echoes some recent discussion in mathematical music theory.[4] Moreover, the geometry of interaction claims that the (logical) *truth* depends on the viewpoint that can be mathematically formalized as a maximal commutative subalgebra $\mathcal{P}$ of the hyperfinite factor II_1. As observed by Jean-Yves Girard [19], a given theorem, for example, may become false with respect to the wrong viewpoint, which means that the viewpoint is part to the *meaning* that one ascribes to the theorem. We believe that this notion of *truth* might reveal some interesting relations with the idea *gestural logic*. This would eventually offer to Twentieth-First Century musicology a new mathematical foundation for a better understanding of the concept of *musical logic*.

[1] In the American Tradition, there are at least two major attempts at proposing music theories as logic-based deductive systems (see [20] and [8]). See also [31] for a discussion of the logical foundation of music theory from the perspective of music analysis.

[2] See [29] for a philosophical discussion about the notion of *musical logic*.

[3] See [19] for a discussion about the logical foundation of quantum physics (and quantum computing) which link the level of proof as *functions* and the level of proof as *actions*.

[4] See [26] for an attempt at discussing the relation between *truth* and *beauty* in music from a mathematical music-theoretical perspective.

References

[1] Amiot, Emmanuel: "Rhythmic Canons and Galois Theory". In: Fripertinger H, Reich L (eds.), "Colloquium on Mathematical Music Theory", *Grazer Math. Ber.*, ISSN 1016-7692, Bericht Nr. 347, 2004, 1-33.

[2] André, Yves "S'orienter dans la pensée mathématique ; l'art des conjectures", Séminaire *Musique, mathématiques et philosophie*, ENS, 10 december 2005. Text available at:
`http://recherche.ircam.fr/equipes/repmus/mamux/YA.pdf`

[3] André, Yves "Aperçus sur les algbres d'oprateurs (algbres de von Neumann)", *École mathmatique pour musiciens et autres non-mathmaticiens*, Séminaire MaMuX/MaMuPhi, Ircam 9 décembre 2006. Transcription and video recording available at:
`http://recherche.ircam.fr/equipes/repmus/mamux/`

[4] André, Yves "Les Topos de Grothendieck", *École mathmatique pour musiciens et autres non-mathmaticiens*, Séminaire MaMuX/MaMuPhi, Ircam 24 March 2006.

[5] Andreatta, Moreno: *Méthodes algébriques en musique et musicologie du XXème siècle : aspects théoriques, analytiques et compositionnels*, PhD thesis, Ircam/EHESS 2003.

[6] Andreatta, Moreno: "On group-theoretical methods applied to music: some compositional and implementational aspects", in Guerino Mazzola, Thomas Noll, Emilio-Luis Puebla ed., *Perspectives in Mathematical and Computational Music Theory*, epOs, Osnabrück 2004.

[7] Assayag, Gérard et al. (eds.): *Mathematics and Music*, Springer Verlag, 2002.

[8] Boretz, Benjamin: *Meta-Variations: Studies in the Foundations of Musical Thought*, PhD Dissertation, Princeton, 1970.

[9] Cartier, Pierre: "A mad day's work: from Grothendieck to Connes and Kontsevich The evolution of concepts of space and symmetry", *Bull. Amer. Math. Soc.* 38, 2001, 389-408. Text available at: `http://www.ams.org`

[10] Connes, Alain: *Noncommutative Geometry*, Academic Press, 1994. Online version available at:
`ftp://ftp.alainconnes.org/book94bigpdf.pdf`

[11] Connes, Alain: "A View of Mathematics", text available at:
`ftp://ftp.alainconnes.org/maths.pdf`

[12] Coven, E. and Meyerowitz, A.: "Tiling the integers with one finite set", *Journal of Algebra*, 212, 1999, 161-174.

[13] Drossos, Costas A: "Sets, Categories and Structuralism". In: Sica, G. (ed.), *What is Category Theory?*, Polimetrica, Monza 2005.

[14] Fuglede, Bent and Kadison R.V.: "On a Conjecture of Murray and von Neumann", *Proc Natl Acad Sci U S A*, 37(7), 1951, 420425.

[15] Fuglede, Bent: "Commuting Self-Adjoint Partial Differential Operators and a Group Theoretic Problem", *J. Func. Anal.*, 16, 1974, 101-121.

[16] Girard, Jean-Yves: *Le Point Aveugle. Cours de théorie de la démonstration*, Roma Tre, Octobre-Décembre 2004. Tome I, *Vers la perfection*, Juin 2006, 296 pp.

[17] Girard, Jean-Yves: "Geometry of interaction III: accomodating the additives", in: J.-Y. Girard, Y. Lafont, L. Regnier (Eds.), *Advances in Linear Logic*, London Mathematical Society Series 222, Cambridge University Press, Cambridge, 1995, pp. 329-389.

[18] Girard, Jean-Yves: "Introduction aux algbres d'opérateurs I : Des espaces de Hilbert aux algbres stellaires", text available at :
`http://recherche.ircam.fr/equipes/repmus/mamux/AlgOpGirard.pdf`

[19] Girard, Jean-Yves "Truth, modality and intersubjectivity", Manuscript, January 2007. Available at:
`http://iml.univ-mrs.fr/ girard/truth.pdf`

[20] Kassler, Michael: *A Trinity of Essays*, PhD thesis, Princeton, 1968.

[21] Mac Lane, Saunders: *Categories for the Working Mathematician*, Springer, Heidelberg et al. 1971

[22] Mac Lane, Saunders and Ieke Moerdijk: *Sheaves in Geometry and Logic*, Springer, New York et al. 1994

[23] Lewin, David: *Generalized Musical Intervals and Transformations*, Yale University Press, New Haven 1987.

[24] Lewin, David: "Klumpenhouwer Networks and Some Isographies that Involve Them", *Music Theory Spectrum*, 12/1, 1990.

[25] Mazzola, Guerino et al.:*The Topos of Music—Geometric Logic of Concepts, Theory, and Performance*, Birkhäuser, Basel et al. 2002.

[26] Mazzola, Guerino: *La Vérité du Beau dans la musique*, forthcoming, Collection "Musique/Sciences", Delatour/France, 2007.

[27] Mazzola, Guerino and Andreatta, Moreno: "From a categorical point of view. K-nets as Limit Denotators", *Perspectives of New Music*, 44(2), Summer 2006, 87-113.

[28] Mazzola, Guerino and Andreatta, Moreno: "Diagrams, gestures and formulæ in music", *Journal of Mathematics and Music*, Vol. 1, No. 1, March 2007, 23-46.

[29] Nicolas, François: "Questions of logic: Writing, Dialectics and Musical Strategies". In: Assayag et al. (eds.): *Mathematics and Music*, Springer Verlag, 2002.

[30] Noll, Thomas: "The Topos of Triads". In: Fripertinger H, Reich L (eds.), "Colloquium on Mathematical Music Theory", *Grazer Math. Ber.*, ISSN 1016-7692, Bericht Nr. 347, 2004, 103-135.

[31] Rahn, John: "Logic, set theory, music theory", in *Music Inside Out. Going too far in musical essays*, Overseas Publisher Association, 2001, 109-125.

[32] Sunder, V. S.: *An Invitation to Von Neumann Algebras*, Springer Verlag, 1987.