The Geometrical Groove: rhythmic canons between Theory, Implementation and Musical Experiment

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Abstract

During the second half of the twentieth century, algebraic methods have been increasingly recognised as powerful approaches to the formalisation of musical structures. This is evident in the American music-theoretical tradition as well as in the European formalised approach to music and musicology. We mention the mathematician and composer Milton Babbitt, the Greek composer Iannis Xenakis and the Roumanian theoretician and composer Anatol Vieru, that gave important impulses to the subject of our paper (Babbitt, 1960; Xenakis, 1971; Vieru, 1980). We also mention Gerald Balzano's original contribution (Balzano, 1980) and Dan Tudor Vuza's model of Vieru's modal theory (Vuza, 1982-), as well the approaches of Guerino Mazzola (Mazzola, 1990), Harald Fripertinger (Fripertinger, 1991) and Marc Chemillier (Chemillier, 1990), who opened the path to a generalisation and implementation of algebraic properties of musical structures.
This paper especially deals with the implementation of Vuza's model of periodic rhythm in OpenMusic, an open source visual language for composition and music analysis developed by IRCAM. This has been done as a part of a specific OM package called Zn, entirely based on the algebraic properties of finite cyclic groups and their applications to music. A complete catalogue of intervallic structures (up to transposition) is the starting point for a classification of such structures by means of musically interesting algebraic properties (Olivier Messiaen's limited transposition property, Milton Babbitt's all-combinatoriality, Anatol Vieru's partitioning modal structures...), their generalisation for any n-tempered system and reinterpretation in the rhythmic domain.
In this article we extend the idea of 'Regular Complementary Canons of Maximal Category' (Andreatta, Agon , Chemillier, 1999) to rhythmic canons of various kinds having the property of tiling musical space (See below).

Rhythmic Canons Tiling the Space

The present essay focuses on the implementation of a family of rhythmic canons having the property of tiling musical time space. Before describing them in terms of an abstract model of cyclic time, we view them as they may appear within a musical composition, in the 'free' linear time, which has no cyclicity. Like in a melodic canon, one has several voices that may enter one after the other until all voices are present. As in the case of a melodic canon all voices are just copies of a ground voice that is suitably translated in the time axis. For simplicity - but yet with respect to linear time - we suppose here, that all voices are extended ad infinitum. We further suppose that the ground voice is a periodic rhythm that we will call the 'inner rhythm'. Following Vuza's definition, a periodic rhythm is an infinite subset R of the rationals Q (marking the attack-points, or onsets) with R = R + d for a suitable period d. Furthermore, R is supposed to be locally finite (i.e. the intersection of R with every time segment [a, b] is finite). The period of a periodic rhythm R is the smallest positive rational number dR satisfying R = R + dR. We also mention another important characteristic of a periodic rhythm - its pulsation pR. It is defined as the greatest common divisor of all distances between its attack points. Obviously, the pulsation pR of a periodic rhythm always divides its period dR.

In order to include the idea of tiling time space into the definition of a rhythmic canon, we need a further preparation: for each voice V of a canon we consider all rational numbers s such that V = R + s. The collection S of all these translations for all voices is itself a periodic rhythm (with period dS dividing dR) that works in fact as a 'meter' for the canon. Note that R and S may have different pulsations pR and pS. The pulsation of a canon is hence to be defined as the greatest common divisor p of the pulsations pR and pS.

The ratio n = dR/p is central in order to switch from linear time, modelled by rational numbers to circular time modelled by residue classes of integers. The transition goes as follows: Let r denote a fixed attack point within the inner rhythm R (If only one canon is being considered one can always suppose r = 0). Then each attack point in any of the voices has the form r + t p for a suitable integer t, i.e. the whole canon is contained in the sublattice r + p Z of Q. Because everything is periodic with period dR, we can work with classes of points in linear time and identify them with cyclic time points. Mathematically, one works with the factor space (r + p Z) / dR Z which may be identified with Z/nZ where n = dR/p.

From now on, we consider the whole canon within Z/nZ. We study the projections of R and S as well as those of the voices V (using the same notation) and formulate additional conditions in order to characterise rhythmic canons. For practical reasons we also allow cycles n that are multiples of dR/p.

Consider two subsets R and S of Z/nZ, the inner rhythm and the outer rhythm. Moreover consider the Voices Vs = R + s, where s runs through S. The pair (R, S) is said to generate a rhythmic tiling canon with the voices Vs if the following conditions are fulfilled:

1) The voices Vs cover entirely the cyclic group Z/nZ. With respect to the linear time this means that the canon is completely tiling musical time space at the (regular) pulsation p.
2) The voices Vs are pairwise disjoint. This means that the voices are complementary.

Periods dR and dS and pulsations pR and pS of R and S are also defined in Z/nZ. Among all canons having the properties 1) and 2) there is the special class of Regular Complementary Canons of Maximal Category, shortly RCMC-Canons (Vuza, 1995). They have the following additional property:

3) The periods pR and pS coincide.

Formally speaking, a RCMC-Canon is a factorisation of a cyclic group Z/nZ into two non periodic subsets (where a subset M of Z/nZ is said to be periodic if there exists an element t in Z/nZ such that t + M = M).

This transition from free linear time to cyclic time, that has been implemented together with all numerical invariants attached to a rhythm (period, number of attacks in a period, pulsation of a rhythm, ...), reduces the difficulty of many operations on rhythms that are connected with the construction of canons. For RCMC-canons, the implementation of Vuza's algorithm on OpenMusic enables to calculate, for any period n, all possible inner and outer structures associated with. It offers to understand the relationships between a period and the number of voices for such a canon. For example, the smallest RCMC-canon has a period equal to 72 and a number of voices equal to 6. This is the consequence of the algebraic property that no cyclic group smaller than Z/72Z can be 'factorised' in two non-periodic subsets. We now use this example to explain the idea of canon modulation.

Canon Modulation

In a compositional situation one might intend to work with more than just a single canon. In that case it is interesting to investigate the inner syntagmatic structures of canons with respect to the paradigmatic relations between several canons. The suggestive term 'canon modulation' shall in fact refer to structural analogies in harmony. A typical modulatory effect in harmony is forced by the re-interpretation of a chord in a new harmonic role.
This works with canons as follows:
Consider a canon consisting of 6 exemplars R + s of the inner rhythm R = (0 1 5 6 12 25 29 36 42 48 49 53) with starting points s in the outer rhythm S = (0 22 38 40 54 56). These starting points s in S parameterise the rhythmic roles of the 6 copies of R within the canon. To modulate into another canon within the same translation class means to modulate into a canon with the same fundamental rhythm R, but S replaced by S + t for some t. Candidates for rhythmical re-interpretation are hence the elements in the intersection of S and S + t. For t = 16 one has S1 = S + 16 = (16 38 54 56 70 0), i.e. there are 4 points in S that can be re-interpreted within S1. The next programm in Openmusic calculates a cyclic sequence of translations of a same outer rhythm S. Note that the number of translation (9) is equal to the lcm of the period and t divided by t.

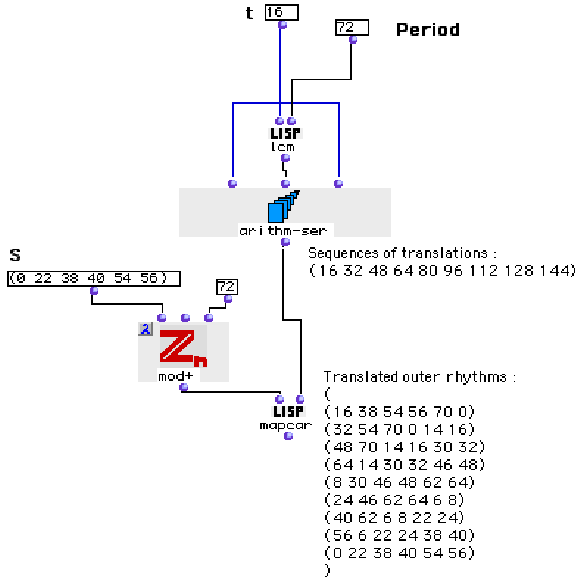

Fig. 1 Openmusic patch showing some translations of a same outer rhythm S. Each translated outer rhythm has 4 elements in common with the previous one.

The example in the figure 2 is constructed as a cyclic sequence of 3 modulations each being a translation of interval 16 mod 72. The lenght of each rhythmic pattern (72 beats) takes up two staffs. We can note that for each modulation two new voices appear replacing two of the six original voices.

Fig. 2. Three modulations of a rhythmic canon with period 72.

Augmented rhythmic canons

While we have been concerned only with translations so far, we will now present a suitable generalisation, that leads to canons with augmented voices. Consider two sequences, R and S, of (invertible affine) symmetries [a,t](x) := a x + t modulo m*n, with R having m entries and S having n. We call R the inner symmetries and S the outer symmetries. Each symmetry consists of an augmentation with factor a and a translation with summand t. Let A(R) and

A(S) denote the augmentation factors of the symmetries in R and S, and let T(R) and T(S) denote the corresponding translations respectively. By R*S we denote the mxn - matrix of all pairwise concatenations r*s where r runs through R and s runs through S. This matrix is a candidate of what we call a "canon of symmetries": By |R*S| we denote the set consisting of all entries in the matrix R*S. Now, the pair (R,S) is said to generate the symmetry canon R*S, if T(|R*S|) = Z/m*nZ, i.e. if every translation mod n*m occurs exactly once among the symmetries in |R*S|. The non-augmented case is characterised through A(R) = (1 ... 1) and A(S) = (1... 1). Moreover, there is a duality of canons in the sense that S*R is the transposed matrix of R*S. In the augmented situation, however, if we are given a symmetry-canon-generating pair (R,S), it is generally not the case that the pair (S,R) is also canon-generating. In case it is, one has T(|R*S|) = T(|S*R|) = Z/m*nZ, but this does not imply |R*S| = |S*R| nor the even stronger condition, that R*S is the transposed matrix (S*R)^ of S*R. From the musical point of view, it is very interesting to make use of the non-commutativity of symmetries, i.e. to benefit from R*S being different from S*R^.
We say, that a symmetry-canon-generating pair (R,S) has a dual one, if |R*S| = |S*R|.

The following is a suggestive example in the case n = 4 and m = 3:
R = ([11, 0] [5, 1] [5, 3] [11 10]) and S = ([11, 0] [5, 1] [5, 5])

In this example one has:

R*S =

[1, 0]	[7, 11]	[7, 7]
[7, 1]	[1, 6]	[1, 2]
[7, 3]	[1, 8]	[1, 4]
[1, 10]	[7, 9]	[7, 5]

(S*R)^ =

[1, 0]	[7, 1]	[7, 5]
[7, 11]	[1, 6]	[1, 10]
[7, 9]	[1, 4]	[1, 8]
[1, 2]	[7, 3]	[7, 7]]

That is, |R*S| = |S*R|, but R*S and (S*R)^ differ from each other in 10 of 12 entries.

Now we explain, how to obtain augmented canons from a symmetry canon. Apply the entries of R to some onset x mod n*m, say x = 0, in order to generate a inner voice. In the example, R*x = R*0 = (0 1 3 10). From R*x one generates the voices of the desired canon by applying the entries of S to R*x. In the example, we obtain the augmented canon:

$$
S*R*0 = \begin{bmatrix} (0 & 11 & 9 & 2) \\ (1 & 6 & 4 & 3) \\ (5 & 10 & 8 & 7) \end{bmatrix}
$$

Analogously, the dual canon turns out to be

$$
R*S*0 = \begin{bmatrix} (0 & 11 & 7) \\ (1 & 6 & 2) \\ (3 & 8 & 4) \\ (10 & 9 & 5 \end{bmatrix}
$$

The composer Tom Johnson experimented with the idea to augment the voices in the sense of playing them at different 'tempos'. (Johnson, 2001). In that case, the lowest common multiple of A(S) defines a long cycle of repetitions of the inner m*n cycle that has to be filled by a suitable number of copies of each voice.

In our example we start with 3 longer voices modulo 4*3*5*11 = 660:

V1 = (0 11 33 110 132 143 165 242 ... 528 539 561 638)
V2 = (1 6 16 51 61 66 76 111 ... 601 606 616 651)
V3 = (5 10 20 55 65 70 80 115 ... 605 610 620 655)

Take, for example, the pattern (0 11 33 110) that has augmentation 11 with respect to (0 1 3 10). In order to create the long voice V1 it must be repeated 5 times. The beginning of the first repetition is 0+12*11=132 and so on.

In order to fill the whole cycle mod 660 one needs 11 copies of V1, 5 copies of V2 as well as 5 copies of V3. Hence, the whole augmented canon consists of 21 voices:
(V1, V1 + 12, ..., V1 + 120, V2, V2 + 12, ..., V1 + 48, V3, V3 + 12, ..., V3 + 48) everything modulo 660. Similarly, its dual canon can be realised with 32 voices.

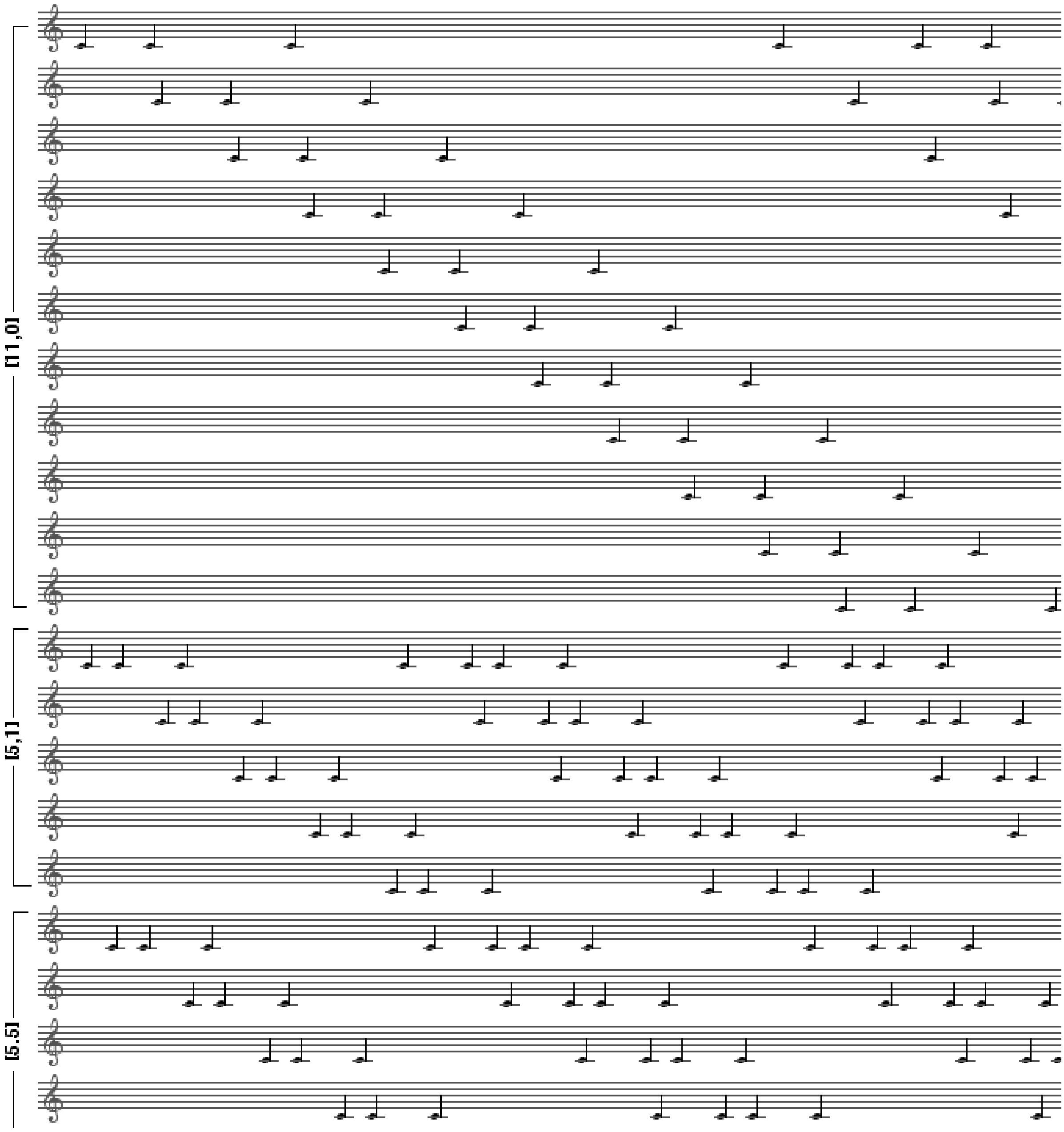

Fig 3 Augmented canon with R = (0 1 3 10) and S = ([11, 0] [5, 1] [5, 5]).

Conclusions

The problem of constructing rhythmic canons tiling the space and the effective possibility to solve it by means of a group-theoretical algorithm shows the usefulness of an algebraic-oriented approach to the formalisation of musical structures. OpenMusic allows the graphical manipulation of rhythmic operations leading to the complete description of two main families of canons that we tried to present in a formal way: the RCMC-canons and the augmented canons. It also shows how to deal with complex musical transformations, as the modulations between different canons. There are suitable OM-Patches (visual programs) in order to produce all examples presented in this paper and to generalise them according with compositional applications or music-theoretical investigations. Positive reactions of composers already working in different compositional projects suggest looking for other musical transformations (like generalised symmetries), including an extension of the model in the pitch domain. This is part of a more general OpenMusic package called Zn that is entirely based on the algebraic properties of (cyclic) groups and their application to music.

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