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Differential geometry applied to acoustics: non linear propagation in Reissner beams

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Although acoustics is one of the disciplines of mechanics, its "geometrization" is still limited to a few areas. As shown in the work on nonlinear propagation in Reissner beams, it seems that an interpretation of the theories of acoustics through the concepts of differential geometry can help to address the non-linear phenomena in their intrinsic qualities. This results in a field of research aimed at establishing and solving dynamic models purged of any artificial nonlinearity by taking advantage of symmetry properties underlying the use of Lie groups. As an illustration, numerical and analytical trajectories of Reisner beams in the configuration space of transformation matrix will be presented.

1 Introduction

The Reissner beam is one of the simplest acoustical system that can be treated in the context of mechanics with symmetry. A Lie group is a mathematical construction that handle the symmetry but it is also a manifold on which a motion can take place. As emphasized by Arnold [1], physical motions of symmetric systems governed by the variational principle of least action correspond to geodesic motions on the corresponding group G. This paper will try, in a first part, to illustrate this basic concept in the case of the continuous group of motion in space. After a literature survey on this subject, an extension from geodesics to auto-parallel submanifolds is proposed in the second part.

2 Nonlinear model for Reissner Beam

2.1 Reissner kinematics

A beam of length L, with cross-sectional area A and mass per unit volume ρ is considered. Following the Reissner kinematics, each section of the beam is supposed to be a rigid body. The beam configuration can be described by a position $\mathbf{r}(s,t)$ and a rotation $\mathbb{R}(s,t)$ of each section. The coordinate s corresponds to the position of the section in the reference configuration Σ_0 (see figure 1).

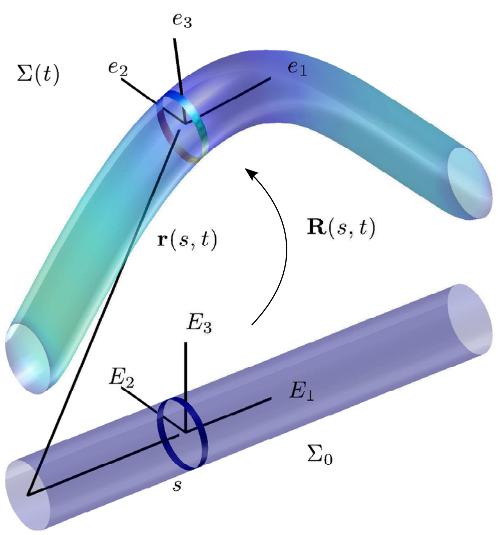

Figure 1: Reference and current configuration of a beam. Each section, located at position s in the reference configuration Σ_0, is parametrized by a translation $\mathbf{r}(s,t)$ and a rotation $\mathbb{R}(s,t) \in SO_3$ in the current configation Σ_t.

2.2 Lie group configuration space

Any material point M of the beam which is located at $\mathbf{x}(s,0) = \mathbf{r}(s,t) + \mathbf{w}_0 = s\mathbf{E}_1 + \mathbf{w}_0$ in the reference configuration ($t = 0$) have a new position (at time t) $\mathbf{x}(s,t) = \mathbf{r}(s,t) + \mathbb{R}(s,t)\mathbf{w}_0$. In other words, the current configuration of the beam Σ_t is completely described by a map

$$\begin{pmatrix}\mathbf{x}(s,t)\\ 1\end{pmatrix} = \underbrace{\begin{pmatrix}\mathbb{R}(s,t) & \mathbf{r}(s,t)\\ 0 & 1\end{pmatrix}}_{\mathbf{H}(s,t)}\begin{pmatrix}\mathbf{w}_0\\ 1\end{pmatrix}, \quad \mathbb{R} \in SO(3), \quad \mathbf{r} \in R^3, \tag{1}$$

where the matrix $\mathbf{H}(s,t)$ is an element of the Lie group $SE(3) = SO(3) \times R^3$, where $SO(3)$ is the group of rotation in R^3. As a consequence, to any motion of the beam a function $\mathbf{H}(s,t)$ of the (scalar) independent variables s and t can be associated. Given some boundary conditions, among all such motions, only a few correspond to physical ones. What are the physical constraints that such motions are subjected to?

In order to formulate those constraints the definition of the Lie algebra is helpful. To every Lie group G, we can associate a Lie algebra $\mathfrak{g}$, whose underlying vector space is the tangent space of G at the identity element, which completely captures the local structure of the group. Concretely, the tangent vectors, $\partial_s\mathbf{H}$ and $\partial_t\mathbf{H}$, to the group $SE(3)$ at the point $\mathbf{H}$, are lifted to the tangent space at the identity e of the group. The definition in general is somewhat technical, but in the case of matrix groups this process is simply a multiplication by the inverse matrix $\mathbf{H}^{-1}$. This operation gives rise to definition of two vectors[1] in $\mathfrak{g} = \mathfrak{se}(3)$

$$\hat{\boldsymbol{\epsilon}}_c(s,t) = \mathbf{H}^{-1}(s,t)\partial_s\mathbf{H}(s,t) \tag{2}$$
$$\hat{\boldsymbol{\chi}}_c(s,t) = \mathbf{H}^{-1}(s,t)\partial_t\mathbf{H}(s,t), \tag{3}$$

which describe the deformations and the velocities of the beam. Assuming a linear stress-strain relation, those definitions allow to define a reduced Langrangian function by the difference of kinetic and potential energy $l(\boldsymbol{\chi}_c, \boldsymbol{\epsilon}_c) = E_c - E_p$, with

$$E_c(\boldsymbol{\chi}_c) = \int_0^L \frac{1}{2}\boldsymbol{\chi}_c^T \mathbb{J}\boldsymbol{\chi}_c ds, \tag{4}$$
$$E_p(\boldsymbol{\epsilon}_c) = \int_0^L \frac{1}{2}(\boldsymbol{\epsilon}_c - \boldsymbol{\epsilon}_0)^T \mathbb{C}(\boldsymbol{\epsilon}_c - \boldsymbol{\epsilon}_0) ds, \tag{5}$$

where $\mathbb{J}$ and $\mathbb{C}$ are matrix of inertia and Hooke tensor respectively and $\hat{\boldsymbol{\epsilon}}_0 = \mathbf{H}^{-1}(s,0)\partial_s\mathbf{H}(s,0)$ correspond to the deformation of the initial configuration.

2.3 Equations of motion

Applying the Hamilton principle to the left invariant Lagrangian l leads to the Euler-Poincaré equation

$$\partial_t\boldsymbol{\pi}_c - ad^*_{\boldsymbol{\chi}_c}\boldsymbol{\pi}_c = \partial_s(\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_0) - ad^*_{\boldsymbol{\epsilon}_c}(\boldsymbol{\sigma}_c - \boldsymbol{\sigma}_0), \tag{6}$$

where $\boldsymbol{\pi}_c = \mathbb{J}\boldsymbol{\chi}_c$ and $\boldsymbol{\sigma}_c = \mathbb{C}\boldsymbol{\epsilon}_c$, (see for example [3], [4] or [5] for details). In order to obtain a well-posed problem,

[1] here, left invariant vector fields

the compatibility condition, obtained by differentiating (2) and (3)

$$\partial_s \chi_c - \partial_t \epsilon_c = ad_{\chi_c} \epsilon_c, \quad (7)$$

must be added to the equation of motion. It should be noted that the operators ad and ad^* in eq. (6)

$$ad^*_{(\omega,\mathbf{v})}(\mathbf{m},\mathbf{p}) = (\mathbf{m}\times\omega + \mathbf{p}\times\mathbf{v}, \mathbf{p}\times\omega) \quad (8)$$
$$ad_{(\omega_1,\mathbf{v}_1)}(\omega_2,\mathbf{v}_2) = (\omega_1\times\omega_2, \omega_1\times\mathbf{v}_2 - \omega_2\times\mathbf{v}_1), \quad (9)$$

depend only on the group $SE(3)$ and not on the choice of the particular "metric" L that has been chosen to described the physical problem [6].

Equations (6) and (7) are written in material (or left invariant) form (c subscript). Spatial (or right invariant) form exist also. In this case, spatial variables (s subscript) are introduced by

$$\hat{\epsilon}_s(s,t) = \partial_s \mathbf{H}(s,t)\mathbf{H}^{-1}(s,t) \quad (10)$$
$$\hat{\chi}_s(s,t) = \partial_t \mathbf{H}(s,t)\mathbf{H}^{-1}(s,t) \quad (11)$$

and (6) leads to the conservation law [18]

$$\partial_t \boldsymbol{\pi}_s = \partial_s(\boldsymbol{\sigma}_s - \boldsymbol{\sigma}_0) \quad (12)$$

where $\boldsymbol{\pi}_s = \mathrm{Ad}^*_{\mathbf{H}^{-1}}\boldsymbol{\pi}_c$ and $\boldsymbol{\sigma}_s = \mathrm{Ad}^*_{\mathbf{H}^{-1}}\boldsymbol{\sigma}_c$. The Ad^* map for $SE(3)$ is

$$\mathrm{Ad}^*_{\mathbf{H}^{-1}}(\mathbf{m},\mathbf{p}) = (\mathbb{R}\mathbf{m} + \mathbf{r}\times\mathbb{R}\mathbf{p}, \mathbb{R}\mathbf{p}). \quad (13)$$

Compatibility condition (7) becomes

$$\partial_s \chi_s - \partial_t \epsilon_s = ad_{\epsilon_s}\chi_s. \quad (14)$$

Equations (6) and (7) (or alternatively (12) and (14)) provide the exact non linear Reissner beam model.

Notations and assumptions vary so much in the literature, it is often difficult to recognize this model (see for example [7] for a formulation using quaternions) . However, this generic statement is used to classify publications according to three axes. In the first one, geometrically exact beam model is the basis for numerical formulations. Starting with the work of Simo [2], special attention is focused on energy and momentum conserving algorithms [8], [9]. Numerical solutions for planar motion are also investigated in [10]. Even, in some special sub-cases (namely where the longitudinal variables do not appear) the non-linear beam model gives rise to linear equations which can be solved by analytical methods [11]. Much of the literature is also devoted to the so-called Kirchhoff's rod model. In this case, shear strain is not taken into account along a thin rod (i.e., its cross-section radius is much smaller than its length and its curvature at all points). In this approximation cross-sections are perpendicular to the central axis of the filament. (see [12] , [14], [15], for example). In that context an interesting geometric correspondence between Kirchhoff rod and Lagrange top can be made [13].

Finally, if only rigid motion is investigated, (i.e. if the spatial dependance in (6) is canceled) the so-called underwater vehicle[2] model is obtained. In absence of exterior force and torque, the equation of motion for a rigid body in an ideal fluid become more simply [16], [17]

$$\partial_t \boldsymbol{\pi}_c = ad^*_{\chi_c}\boldsymbol{\pi}_c, \text{ that is } \begin{cases} \dot{\mathbf{m}} = \mathbf{m}\times\omega + \mathbf{p}\times\mathbf{v} \\ \dot{\mathbf{p}} = n\times\omega \end{cases} \quad (15)$$

[2]underwater vehicle in the case that the center of buoyancy and the center of gravity are coincident

In this simpler form, a geometric interpretation is easier. The solution of the equation of motion mentioned above, if it exists, should be interpreted as a geodesic of the group $SE(3)$ endowed with a non-canonical left invariant metric $\mathbb{J}$. To accomplish the correspondence between the Euler-Poincaré's equation and geodesic equation the historical definition of the covariant derivative is exposed in the next section.

3 Geometric interpretation

3.1 Geodesics on curved spaces

A trajectory of a particle of mass m which is moving on a manifold[3] M can be thought as a curve $\alpha(t)$ on M and $\mathbf{v}(t) = \dot{\alpha}(t)$ is the speed of the particle. According to the Newton's second Law of motion, its acceleration (the variation of its velocity) is proportional to the net force acting upon it $\sum \mathbf{F} = m\frac{d\mathbf{v}}{dt}$. The expression of this variation, $\mathbf{v}(t+dt) - \mathbf{v}(t)$, shows that the velocities are evaluated at two different points of the curve: $\alpha(t+dt)$ and $\alpha(t)$ which are, *a priori*, incommensurable quantities. So, one of the two vectors needs to

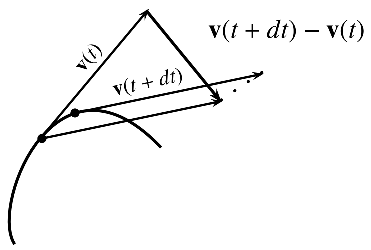

Figure 2: For flat manifolds, a trivial parallel transport is used to compute the acceleration.

be parallel transport as it is illustrated, for flat manifolds, in figure (2). For curved manifolds the operation is not so easy and its historical construction is related by M.P. do Carmo in [19] for surfaces of R^3.

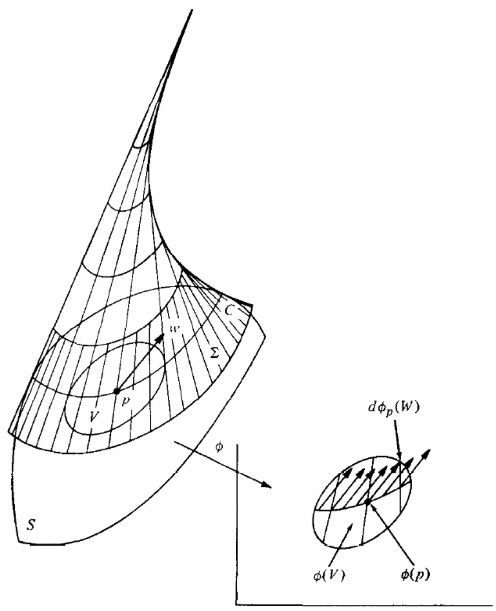

Figure 3: Parallel transport along a curve

[3]a surface for short

Let $\alpha(t)$ be a curve on a surface S and consider the envelope of the family of tangent planes of S along α (see figure 3). This envelope[4] is a regular surface Σ which is tangent to S along α. Thus, the parallel transport along α of any vector $\mathbf{w} \in T_p(S)$, $p \in S$, is the same whether we consider it relative to S or to Σ. Furthermore, Σ is a developable surface; hence can be mapped by an isometry ϕ into a plane P (without stretching or tearing). Parallel transport of a vector $\mathbf{w}$ is then obtained using usual parallel transport in the plane along $\phi(\alpha)$ and pull it back to Σ (by $d\varphi^{-1}$).

Technically, this historical construction gives rise to the concept of the covariant derivative $\frac{D\mathbf{w}}{dt} = \nabla_{\mathbf{v}}\mathbf{w}$ of a vector field $\mathbf{w}$ along α. The parametrized curves $\alpha : I \to R^2$ of a plane along which the field of their tangent vector $\mathbf{v}(t)$ is parallel are precisely the straight lines of that plane. The curves that satisfy an analogous condition , i.e.

$$\frac{D\mathbf{v}}{dt} = \nabla_{\mathbf{v}}\mathbf{v} = 0, \qquad (16)$$

for a surface are called geodesics. Intuitively, the acceleration as seen from the surface vanishes : in absence of net force, the particle goes neither left nor right, but straight ahead.

The kinetic energy (4) define a left invariant Riemannian metric on $SE(3)$, and then define also a symmetric connexion ∇ which is compatible with this metric (Levi-Civita connexion). It can be shown that geodesic equation (16) for this particular connexion coincide with Euler-Poincaré equation of motion (15) when $SE(3)$ is endowed with the kinetic metric (4).

Now, this equation deals with motion of rigid body described by a single scalar variable t. So what is the geometric interpretation of the equations of motion (6) and (7) where two variables s and t are involved. In other word, can we extend a geodesic, which is a 1-dimensional manifold, to 2-dimensional geodesic ?

3.2 Auto-Parallel submanifolds

A geodesic curve on a surface S is a 1-dimensional submanifold of S for which the parallel transport of its initial velocity stay in its own tangent space. In that sense, a geodesic is an auto-parallel curve. If now, geodesics are seen as auto-parallel curves on surface, a definition of an n-dimensional auto-parallel submanifolds can be made.

A submanifold M is auto-parallel in S if the parallel translation of any tangent vector of M along any curve in M stays in its own tangent space $T(M)$. It should be notice, that a parallel translation of a vector $\mathbf{w} \in T(M)$ certainly belongs to $T(S)$ but not necessarily $T(M)$. In orther words, M is auto-parallel in S with respect to the connection ∇ of S if

$$\nabla_{\mathbf{X}}\mathbf{Y} \in T(M), \quad \forall \mathbf{X}, \mathbf{Y} \in T(M) \qquad (17)$$

A correspondence between auto-parallel surface and solutions to equations (6) and (7) is still to be demonstrated. But if it is the case, any motion of the beam could then be seen as an auto-parallel surface Σ immersed in the group $G = SE(3)$

$$\begin{aligned} \mathbf{H} \quad : \quad & A = [0, L] \times R \in R^2 \to SE(3) \\ & (s, t) \to \mathbf{H}(s, t), \end{aligned}$$

[4]Assume that $\alpha(t)$ is nowhere tangent to an asymptotic direction

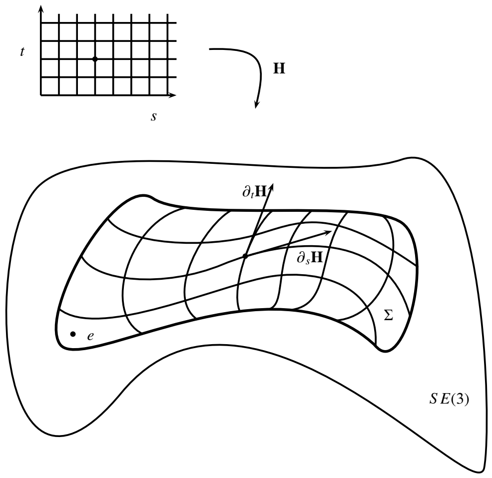

Figure 4: Symbolic representation of a parametrized surface Σ immersed into the group $G = SE(3)$

rather than a function of two variables as it is illustrated symbolically in figure (4). In this perspective, solving a physical variational problem is therefore transposed to the problem of finding a immersed surface which is auto-parallel but, since the parametrization is far from being unique, finding also the particular system of coordinates that has a physical meaning.

4 Conclusion

In the literature, it seems that auto-parallel submanifolds coincide with totally geodesic submanifolds [20]. This gives some mathematical tools to treated physical problems in a more general form. In particular, the non linear normal modes can be revealed to be auto-parallel surfaces with periodic boundary conditions.

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