19th INTERNATIONAL CONGRESS ON ACOUSTICS
MADRID, 2–7 SEPTEMBER 2007

THREE-DIMENSIONAL INTERACTION BETWEEN STRINGS, BRIDGE AND SOUNDBOARD IN MODERN PIANO'S TREBLE RANGE

PACS 43.75.Mn

Cuenca, Jacques [1] ; Caussé, René [2]

Institut de Recherche et Coordination Acoustique/Musique (IRCAM), UMR CNRS 9912;
1 place Igor Stravinsky, 75004 Paris, France
[1] Jacques.Cuenca.Etu@univ-lemans.fr
[2] causse@ircam.fr

ABSTRACT

A model of the vibrational coupling between the strings, the bridge and the soundboard of the piano in its treble register is presented. Each one of these vibrating structures is separately described by its mechanical input admittance matrix in which the nine terms represent impulse responses or frequency responses to a vectorial input force. The total admittance matrix is analytical and is calculated by assembling the latter in accordance with classical mechanical continuity equations. This formalism allows to study the role of each part of the system by simulating the interaction between a chosen number of structures. Simulations show that the two perpendicular polarisations of bending motion of a single string influence each other due to the elastic response of the bridge. Similar to this is the coupling between unison strings. These interactions give rise to beats and double decay phenomena, which are of particular interest to professional piano makers and tuners. The influence of sound radiation by the soundboard and of duplex strings on the temporal behaviour of the piano and on the spectral content of vibrations is discussed as well. Simulations will be presented and discussed comparing them to experimental work carried on a simplified piano.

INTRODUCTION

One of the main stages of sound production on stringed musical instruments is the transmission of vibrational energy from strings to the radiating part of the instrument. The string's end supports are not completely rigid and allow an interaction between the string and the body of the instrument that can be very strong. Parameters of the body such as wood properties or geometry can influence noticeably the oscillatory behavior of the string. In the treble register of the piano, the interaction between the strings, the bridge and the soundboard plays a crucial role since it is responsible for the so-called "double-decay" behavior of a unison string doublet or triplet. This topic has attracted the attention of both scientific and musical communities regarding the middle range of the instrument. However, the double decay plays a determinant role in the treble range because sound level is lower and decay time is shorter than in the rest of the instrument. In general, piano makers affirm that these two parameters need to be enhanced.

The double-decay behavior can arise as a consequence of two different phenomena that are both related to vibration transmission through the bridge. Weinreich [1] showed experimentally and by a simple physical model that the two or three strings act as simple one-degree-of-freedom oscillators in which slight relative mistuning leads to a first "in-phase" fast decay with a large amount of energy transferred to the soundboard. This is followed by an "out-of-phase" slow decay, or "aftersound", in which vibrational energy is mostly transferred between the strings, and losses through the soundboard are reduced. The second phenomenon resulting in double decay is the interaction between the two perpendicular polarisations of bending motion of a single string via the three-dimensional elastic response of the bridge. After the hammer strikes the string vertically, a large amount of energy is initially dissipated by the soundboard. Simultaneously, the three-dimensional strain-displacement relations within the bridge allow horizontal motion of the strings to be gradually excited. Horizontal polarisation lasts longer because of the poor sound radiating efficiency of the soundboard in this direction.

Another important aspect in the treble range of some piano is the existence of duplex scale strings. Each string has two parts: a "speaking length" and a "duplex length". The latter is not directly struck by the hammer and is set into motion by sympathetic excitation by mechanical coupling via the bridge. In the bass and middle registers of the instrument, duplex strings are damped with a felt ribbon. However, in the treble range they are used by piano makers to enhance vibrating level or to enrich the strings' speaking length vibration spectrum. Please refer to the communication [2] also presented in this congress for more details about this subject.

The aim of this work is to describe the mechanical contact between strings, bridge and soundboard in the treble register of the piano in order to obtain an analytical expression for the mobility of the coupled system in time or frequency. Each one of these vibrating structures is described by an admittance matrix in which each term is an elementary velocity response to an excitation in another direction. The total admittance matrix is obtained by assembling individual matrices according to classical mechanical continuity relations. The formalism allows to study separately the influence of the bridge, the soundboard and a duplex string on the vibrations of the whole system.

ANALYTICAL MODEL

Formalism

Each structure in the system (strings, bridge and soundboard), is described by an input admittance matrix $\mathbf{Y}_n(t)$, that characterizes its mobility in the three directions of space mobility. The terms or $Y_{ij}(t)$ are then understood as velocities along the i-direction per unit force applied along the j-direction, where $i, j = x, y, z$. Assuming that the transverse dimensions of the bridge are much smaller than the wavelength in wood, for each note of the instrument the strings and the soundboard can be considered as being coupled to the bridge at a single point. Therefore, all the structures have the same velocity at this point and each one of them is excited by a different force. In the general case, for N structures:

$$\vec{v}_n = \vec{v}_m \;\text{ and }\; \vec{f}_{tot} = \sum_{n=1}^{N} \vec{f}_n \,, \tag{1}$$

where $n, m = 1, ..., N$. The simplified system can be represented as a three-dimensional diagram or as an equivalent electrical circuit (figure 1).

In the frequency domain, the total admittance matrix is given by [3]:

$$\hat{\mathbf{Y}}(\omega)^{-1} = \sum_{n}^{N} \hat{\mathbf{Y}}_n(\omega)^{-1}, \tag{2}$$

where ' $\hat{}$ ' denotes Fourier transform. The coupled system responds to a total excitation $\hat{\vec{f}}$ with the velocity

$$\hat{\vec{v}} = \hat{\mathbf{Y}} \cdot \hat{\vec{f}}. \tag{3}$$

In the time domain, this relation takes the form of the convolution product $\vec{v} = \mathbf{Y} * \vec{f}$.

Admittance matrix for each part of the system

1. String

On a traditionnal piano, each string has several parts of different lengths, delimited by the sounding post and the bridges. Only the two lengths separated by the bridge situated on the soundboard are of significative importance: one of them is struck by the hammer and the other vibrates by sympathetic excitation via the bridge. For the bridge, these two parts of the string are equivalent and the interactions between them are symmetrical.

Transverse waves in the string are described by the classic equation [4]:

$$\mu\frac{\partial^2 u}{\partial t^2} - T\frac{\partial^2 u}{\partial x^2} + EI\frac{\partial^4 u}{\partial x^4} + \mu\frac{\omega}{Q}\frac{\partial u}{\partial t} = 0, \tag{4}$$

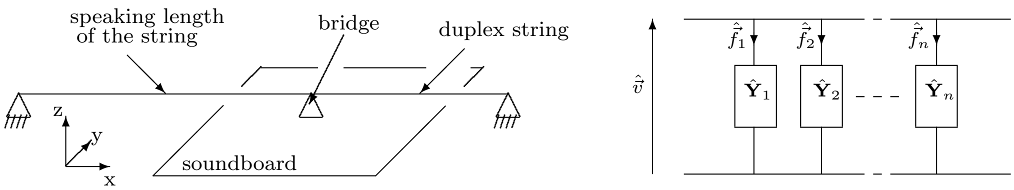


FIG. 1: *Simplified mechanical system and electrical representation at the bridge.*

where $u = u(x;t)$ is the transverse dispacement of an infinitesimal portion of the string, μ its mass per unit length, T its static tension, E its Young's modulus, I its quadratic geometrical momentum and Q its quality factor.

The boundary conditions can be written as:

$$\begin{cases} -T\frac{\partial u}{\partial x}(x=0,t) = f^{(ext\to s)}(t), \\ u(x=L,t) = 0. \end{cases} \quad (5)$$

where $f^{(ext\to s)}(t)$ is the total external force applied to the string. Assuming that the string is perfectly cylindrical, equations 4 and 5 apply for the y and z directions and lead to the admittance matrix:

$$\hat{\mathbf{Y}}^{(\mathbf{s})}(\omega) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & j\frac{\omega}{Tk}\tan(kL) & 0 \\ 0 & 0 & j\frac{\omega}{Tk}\tan(kL) \end{bmatrix}, \quad (6)$$

where the wave number k is given by the dispersion equation: $EIk^4 + Tk^2 - \mu\omega^2 + j\frac{\omega^2\mu}{Q} = 0$, obtained by Fourier transform of equation 4. The admittance matrix of the string is diagonal, which means that its two polarisations of transverse motion are independant from each other.

2. Bridge

The bridge's dimensions are not significant enough to take into account wave propagation between strings and soundboard. Therefore, it is useful to approximate the bridge by an infinite medium in which a single point is excited by the string terminations and the soundboard. This leads to a characteristic admittance matrix for the bridge (no boundaries are present), thus underestimating its mobility. Moreover, this hypothesis requires the definition of a contact surface S with strings and soundboard, which can be estimated experimentally.

The equation of motion of a conservative elastic three-dimensional infinite medium can be written in tensorial notation as [5]:

$$\rho\frac{\partial^2 u_i}{\partial t^2} - C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0, \quad (7)$$

where ρ is the density of wood, u_i is the displacement vector and C_{ijkl} is the orthotropic stiffness tensor, defined as [6]:

$$C_{IJ} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}. \quad (8)$$

Strings and soundboard apply a stress $\tau_i^{(ext\to b)}$ to the bridge at the point $x = 0$. The corresponding Newton's law can be written as:

$$\sigma_{ij}\cdot N_j = \tau_i^{(ext\to b)}, \quad (9)$$

where σ_{ij} is the stress tensor, N_j is the normal unitary vector to the contact surface S and $\tau_i^{(ext\to b)}$ is the vector of external stresses applied to the bridge.

As the string's longitudinal vibrations are not taken into account, the motion of the bridge can be simplified to a propagation in the (y,z) plane, perpendicular to the main axis of the string. Combining equations 7 and 9 leads to:

$$\hat{\mathbf{Y}}^{(b)}(\omega) = \frac{j\omega/S}{D}\begin{bmatrix} Y_{xx} & 0 & 0 \\ 0 & Y_{yy} & Y_{yz} \\ 0 & Y_{zy} & Y_{zz} \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned} &D = kC_{55}C_{44}n_3\left(n_3^2C_{33} - n_2^2C_{23}\right), \\ &Y_{11} = C_{44}(C_{33}n_3^2 - C_{23}n_2^2), \\ &Y_{22} = C_{55}C_{33}n_3^2, \\ &Y_{23} = -C_{44}C_{55}n_2n_3, \\ &Y_{32} = -C_{23}C_{55}n_2n_3, \\ &Y_{33} = C_{44}C_{55}n_3^2. \end{aligned}$$

The wave-number vector is defined as $k_i = k\cdot n_i$, with its modulus k and a normalised vector n_i.

3. Soundboard

The role of the soundboard is to convert vibrating energy into sound radiation by contact between its surface and the sourrounding fluid. This phenomenon is mainly related to bending motion perpendicular to its surface. In order to obtain analytical expressions for the displacement field and the admittance of the soundboard, the latter is modelled by a rectangular anisotropic plate, of which parameters and elastic constants such as density ρ_p, thickness h, Young's moduli E_x and E_y and Poisson's ratio ν_{ij} are chosen according to conventional spruce soundboards. The bending-motion equation in the z-direction for an anisotropic plate is [7]:

$$D_x \frac{\partial^4 u_z}{\partial x^4} + D_y \frac{\partial^4 u_z}{\partial y^4} + \rho_p h \frac{\partial^2 u_z}{\partial t^2} = \delta_{(x-x_b)}\, \delta_{(y-y_b)}\, \sigma_z + p, \tag{11}$$

where $D_x = \dfrac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})}$ and $D_y = \dfrac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})}$ are the flexural rigidities of the plate along the x and y axes, respectively. Besides, $\sigma_z = \sigma_z^{(ext \to p)}$ is the total external stress applied to the plate and $p = p(x, y, z = 0; t)$ represents total acoustic pressure on both sides of the plate. Boundary conditions for simply supported edges can be written:

$$\forall (x, y) \in \partial S_p, \qquad u_z(x, y; t) = 0. \tag{12}$$

The radiated pressure is linked to the plate's velocity by the Euler's equation $\dfrac{\partial p}{\partial z} = -\rho_{air} \dfrac{\partial^2 u_z}{\partial t^2}(x, y; t)$, which in the frequency domain takes the form:

$$\hat{p}(x, y, 0; \omega) = j \frac{\rho_a \omega^2}{\kappa_z} \hat{u}_z(x, y; \omega), \tag{13}$$

where ρ_a and κ_z are respectively air density and wave number in air. Equation 13 can often be found in a simplified modal form in which ω is replaced by the modal angular frequencies ω_{mn}, assuming that sound radiation occurs only at resonance frequencies.

The method for describing the behavior of the soundboard in the treble range of the piano is taken from [8] and consists on calculating an asymptotic continuous admittance in high frequencies (or for a high modal density of the plate) from a classic discrete modal description. By writting the plate's displacement as the modal sum:

$$\hat{u}_z(x, y; \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}(\omega) \Phi_{mn}(x, y), \tag{14}$$

equations 11 and 13 lead to the discrete admittance:

$$\hat{Y}^{(p)}(x_b, y_b; \omega) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{j\omega/S \;\; \Phi^2_{mn}(x_b, y_b)}{\left(\rho_p h (\omega^2_{mn} - \omega^2) - j \frac{\rho_a}{\kappa_z} \omega^2 \right)}. \tag{15}$$

For a plate that expounds a high modal density, the admittance becomes:

$$\hat{Y}^{(p)}(x_b, y_b; \omega) = \int_0^{\infty} \frac{j\omega/S \;\; \Phi^2_{mn}(x_b, y_b)\, \chi\, d\omega_{mn}}{\left(\rho_p h (\omega^2_{mn} - \omega^2) - j \frac{\rho_a}{\kappa_z} \omega^2 \right)}, \tag{16}$$

where $\chi = \frac{dm dn}{d\omega_{mn}}$ is defined as modal density (or number of modes per unit frequency), in seconds. Eigenfrequencies ω_{mn} become a continuous variable and the modal shape Φ_{mn} can be considered constant. The integral is then evaluated by contour integration, leading to the mean-value admittance of the plate in the z-direction:

$$\hat{Y}^{(p)} = \frac{\pi\, \chi}{2\, S\, \rho_p\, h}. \tag{17}$$

The admittance matrix can then be represented in the form:

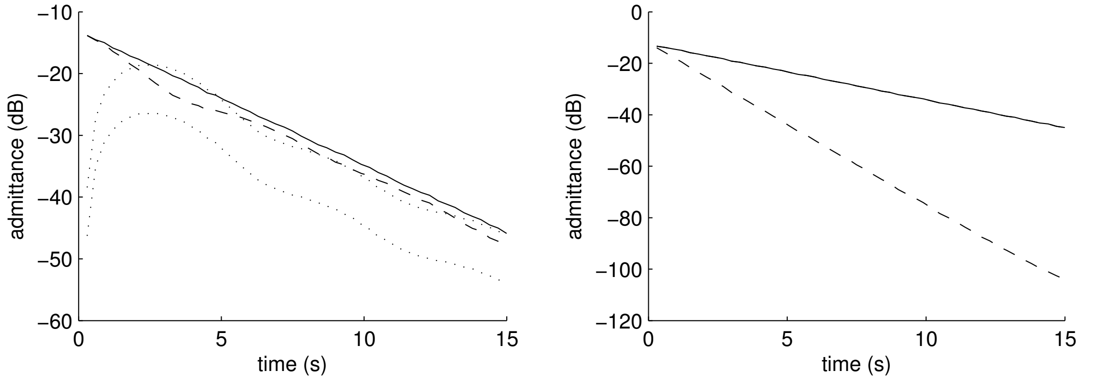
$$\hat{\mathbf{Y}}^{(p)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hat{Y}^{(p)} \end{bmatrix}. \tag{18}$$

NUMERICAL SIMULATIONS

Once the entire system admittance matrix calculated from equation 2, the various terms can be plotted as spectra or as impulse responses. In order to underline the effect of each part of the system, simulations are first done on simple assemblies, then on more complex systems. The simulations are carried on a $D_4^{\#}$ string, which was used by Weinreich [1], though the corresponding wavelength reaches the limits of the approximations undertaken for the elaboration of this method.

Coupling between string's orthogonal bending polarisations

The system here simulated is the contact point between the string, the bridge and the soundboard. The admittance matrix of the coupled system is then observed as a function of time at the connection point between the two structures and the various terms are compared to the response of an uncoupled string (figure 2).



(a) *Impulse responses of a $D_4^{\#}$ string connected to the bridge. Solid: $Y_{yy}^s(t)$ or $Y_{zz}^s(t)$ (string alone); dashed: $Y_{yy}(t)$ or $Y_{zz}(t)$; dotted: $Y_{zy}(t)$ (high) and $Y_{yz}(t)$ (low)*

(b) *Impulse responses of a $D_4^{\#}$ string connected to the soundboard. Solid: $Y_{yy}^s(t)$ or $Y_{zz}^s(t)$ (string alone); dashed: $Y_{zz}(t)$ (coupled system)*

FIG. 2: *Effect of the bridge and the soundboard in the string's response*

Figure 2(a) shows that the string has a constant decay rate by itself but, in the presence of the bridge, its bending motion polarisations interact between theirselves: a complementarity exists between direct (Y_{yy} or Y_{zz}) and cross (Y_{yz} or Y_{zy}) responses. This means that vibratory energy is allowed to swing amongst the two bending motion polarisations of the string, in the form of a beat phenomenon. Furthermore, it is possible to observe that direct responses are equal in the y and z directions, while cross responses differ from each other by a translation of nearly $8\,dB$, due to the intrinsic non-symmetry of bridge's admittance matrix.

In order to bring relief to the soundboard's role, the bridge is now replaced by the soundboard and the same kind of simulation is carried out. The responses of the string and of the compound system at the connection point are superimposed on figure 2(b). It can be observed that the soundboard acts as an energy dissipator, thus increasing the decay rate of oscillations in the z-direction, without affecting further response characteristics.

When the soundboard is added to the string-bridge system, the decay of vibrations splits into two phases: in the first, the string vibrates vertically and thus communicates a large amount of energy to the soundboard (figure 3). The horizontal polarisation is progressively excited through elastic response of the bridge and lead to a second slow decay. This is often called the "double-decay" behavior of piano tones, which is observable in radiated sound pressure as well [1]. It is important to note that the soundboard makes the beats dissapear since the decay rates for the various polarisations differ.

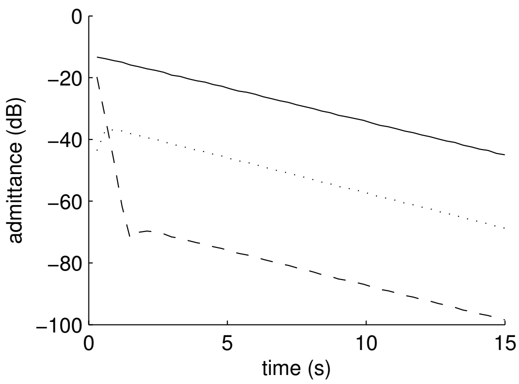


FIG. 3: *Impulse responses of a $D_4^{\#}$ string connected to the bridge and the soundboard. Solid: $Y_{yy}^s(t)$ or $Y_{zz}^s(t)$ (string alone direct response); dashed: $Y_{zz}(t)$ (compound system direct response) ; dotted: $Y_{zy}(t)$ (compound system cross response).*

Coupling between unison strings

It is well known that professional piano tuners set the two or three strings of a note in a slightly "mistuned" configuration in order to strike a balance between attack and aftersound. Differences in fundamental frequency between strings in a such tuned piano are situated in the range of 0.5 to 4 cents, yet the frequency differences are not perceived as a mistuning since human ear is only sensitive to frequency differences or variations greater than 8 cents, approximately. Figure 4 shows the double decay in impulse responses of the mechanical system composed by two strings, the bridge and the soundboard. Simulations with 2 and 4 cents mistunings are superimposed for comparison. In particular, it is possible to remark that the higher the tuning discrepancy, the shorter the first decay phase and thus the higher the vibratory level in the second decay.

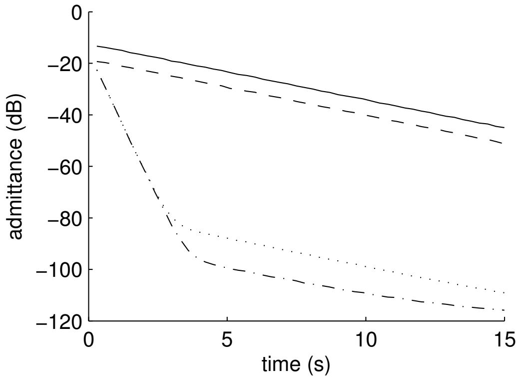


FIG. 4: *Impulse responses of two $D_4^\#$ strings connected to the soundboard for two different relative mistuning amounts. Solid: $Y_{yy}^s(t)$ or $Y_{zz}^s(t)$ (single string direct response); dashed: $Y_{yy}(t)$: compound system direct response in y-direction; dashed-dotted: $Y_{zz}(t)$: direct response in the z-direction for a 2-cents mistuning; dotted: $Y_{zz}(t)$: direct response in the z-direction for a 4-cents mistuning.*

Influence of duplex string

Piano makers believe that the portion of string remaining between the bridge and the hitch pin can be useful to enhance note duration if tuned at a fifth, an octave or at unison from the speaking length. In fact, duplex strings perturb the spectral content of single-string vibrations by enriching high partials. Meanwhile, in the present model the two strings are equivalent since the system is excited and observed at the bridge. In reality, the duplex string is excited through the bridge from vibrations of the speaking length at its particular frequencies. The foregoing considerations about mistuning are then to be applied to common partials between both parts of the string.

CONCLUSIONS

Simulations show that the bridge induces an interdependence between the two polarisations of bending motion of piano strings. The double-decay behavior of oscillations at the bridge arises as a consequence of wood properties and soundboard geometry or of slight tuning discrepancies among unison strings. The possibility to balance attack and aftersound corroborates piano makers and tuners' knowledge. Furthermore, the soundboard appears to act as a simple energy dissipator in its perpendicular direction and the duplex length of the string perturbs the spectral content of the vibrations of the speaking length. Moreover, the model here proposed is entirely analytic and the formalism is of great generality since it allows to assemble several mechanical structures in order to compute the coupled system's response. A simplified piano is under construction at Ircam for experimental validation of the present work.

References

1. G. Weinreich, *"Coupled piano strings"*, J. Acoust. Soc. Am., **62**(6), pp 1474-1484, (1977).
2. P. Zelmar, R. Caussé, *"Experimental and theoretical study of the vibration of strings in the high register of the piano. The effect of the 'duplex scale' "*, 19th International Congress on acoustics, Madrid, 2007.
3. J. Woodhouse, *"On the Synthesis of Guitar Plucks"*, Acustica - Acta Acustica, **90**, pp 928-944, (2004).
4. C. Valette, C. Cuesta, *"Mécanique de la Corde Vibrante"*, Hermès, 1993 (in french).
5. E. Dieulesaint, L. Royer, *"Ondes Élastiques dans les Solides"*, Masson, 1997 (in french).
6. V. Bucur, *"Acoustics of wood"*, CRC Press, 1995.
7. C. Lesueur, *"Rayonnement Acoustique des Structures"*, Eyrolles, 1988 (in french).
8. E. Skudrzyk, *"The mean-value method of predicting the dynamic response of complex vibrators"*, J. Acoust. Soc. Am., **67**(4), pp 1105-1135, (1980).