Wind instruments as time delay systems. Part I : modeling ⋆

B. d'Andréa-Novel * **J.-M. Coron** ** **B. Fabre** *** **T. Hélie** ****

* *Mines-ParisTech, CAOR, 60 bvd Saint-Michel, 75272 Paris Cedex 06, FRANCE (e-mail: brigitte.dandrea-novel@mines-paristech.fr)*
** *UPMC and IUF, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, FRANCE (e-mail: coron@ann.jussieu.fr)*
*** *UPMC, Lutherie, Acoustique, Musique, IJLRA, 11 rue de Lourmel, 75015 Paris, FRANCE (e-mail: benoit.fabre@upmc.fr)*
**** *IRCAM-CNRS UMR 9912, 1 place Igor Stravinsky, 75004 Paris, FRANCE (e-mail: thomas.helie@ircam.fr)*

Abstract: In this paper wind instruments are modeled as time delay systems. In fact, a wind instrument is usually made of a linear acoustic resonator (the pipe) coupled with a nonlinear oscillator (the mouth of the instrument). The resonator can be modeled through hyperbolic wave equations. Two kinds of instruments are considered : the **first** one is a **slide flute**, (i.e. a kind of recorder without finger holes but ended by a piston to modify the pipe length), for which a realistic model of the air jet coupled with the pipe is given. The **second example** will concern the case of a **simplified trumpet-like instrument** composed of a valve (including the mechanics of the lips, contrary to the case of the flute), an air jet coupled with the valve dynamics and an acoustic pipe excited by the jet and radiating in the air. The overall system can be described by a so-called **nonlinear neutral state space system**

Keywords: Wind instruments; Hyperbolic PDEs; Boundary conditions; Neutral delay system.

1. INTRODUCTION

Many physical models of musical instruments are available in the literature (see for example [Fletcher, Rossing(1998)], [Chaigne, Kergomard(2008)]). But obtaining realistic musical rendering is usually a problem, especially for self-sustained instruments. Realistic physical models for analysis and synthesis for flue musical instruments such as organs or recorders has been an important research subject for a few decades. We will not be exhaustive, but we can mention paper [Cremer, Ising(1968)] giving a first quasi-stationary model of the jet drive, which has been later improved by many authors (see e.g. [Fletcher(1976)]). The works of [Howe(1975)] pointed out the importance of vortex shedding at the labium. In fact, in steady blowing conditions, models not taking into account this effect (e.g. in [Fletcher(1976)]) led to an overestimation of the amplitude of the pressure oscillation in the pipe. Therefore, as in [Verge, Caussé et al.(1994)], we have taken into account these interactions jet/labium, but as already mentioned, the system we are studying is different: the resonator's length is time-varying, controlled through the piston mechanism and there is no finger hole. The whole structure can then be described by two linear PDEs coupled with nonlinear ODEs describing the boundary conditions:

- for the mouth, taking into account the jet dynamics,
- and for the piston.

We will also consider another kind of wind instrument : a **simplified trumpet-like instrument** which, contrary

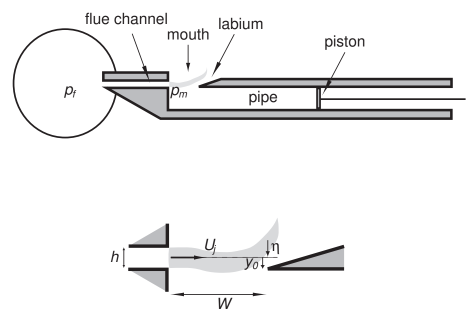

Fig. 1. The slide flute

to the case of the flute, is composed of a valve including the mechanics of the lips. The lips vibrations modulate the air inflow which will excite the acoustic wave in the pipe and then radiate in the air at the output of the resonator (see Fig.2).

The structure of the paper is as follows : in section 2 we recall our pipe model. In section 3 we give the physical models of the jet channel and the mouth in the case of the two instruments. Section 4 is devoted to the computation of the boundary conditions at the end of the resonator and at its entrance as well as a state-space representation for both instruments.

2. PHYSICAL MODEL OF THE PIPE

If ρ_0 denotes the fluid (here the air) density at rest, S_p the constant section of the pipe which is supposed to be cylindrical, and assuming the flow rate $u(x,t)$ at time t

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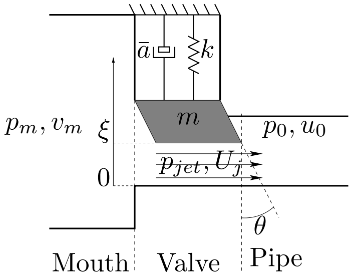

Fig. 2. The dynamics of the musician's lips is modeled by that of a solid mass subjected to pressure forces, a damping, and a spring.

and point x in the pipe and the relative pressure $p(x,t) = P - P_{atm}$ (P_{atm} denoting the atmospheric pressure) are uniform on a section, the Euler equation, giving the fluid dynamical properties can be written:

$$\frac{\partial u}{\partial t} = -\frac{S_p}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

neglecting the viscous and thermal effects near the walls.

The mass conservation law has the following form :

$$\frac{\partial \rho}{\partial t} = -\frac{\rho_0}{S_p}\frac{\partial u}{\partial x}. \tag{2}$$

Finally, assuming that the transformation is adiabatic, we have the following equation, c being the sound velocity in the fluid :

$$p = c^2 \rho \tag{3}$$

which allows to link the relative pressure and the density of the fluid in which the acoustical wave evolves.

Then, replacing ρ from p in (2), we obtain the second state equation which completes (1), i.e.:

$$\frac{\partial p}{\partial t} = -\frac{\rho_0 c^2}{S_p}\frac{\partial u}{\partial x}. \tag{4}$$

Differentiating (4) with respect to t, (1) with respect to x and collecting the resulting equations lead to the d'Alembert equation :

$$\frac{\partial^2 p}{\partial t^2} - c^2\frac{\partial^2 p}{\partial x^2} = 0. \tag{5}$$

Equations (1) and (4) allow to write the system dynamics in the following state-space form with $X = \begin{pmatrix} u \\ p \end{pmatrix}$:

$$\frac{\partial X}{\partial t} + A\frac{\partial X}{\partial x} = 0\,, \text{ with } A = \begin{pmatrix} 0 & S_p/\rho_0 \\ \rho_0 c^2/S_p & 0 \end{pmatrix}. \tag{6}$$

This representation can be diagonalized :

$$\partial_t Z + \Lambda \partial_x Z = 0\,, \text{ with } \Lambda = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \tag{7}$$

where the change of coordinates is given by :

$$Z = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} u + \dfrac{S_p}{\rho_0 c}p \\ u - \dfrac{S_p}{\rho_0 c}p \end{pmatrix} \tag{8}$$

and

$$X = \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha+\beta}{2} \\ \dfrac{\rho_0 c(\alpha-\beta)}{2S_p} \end{pmatrix}. \tag{9}$$

The eigenvalues $c > 0$ and $-c < 0$ being respectively the velocity of the ingoing wave $\alpha(x,t)$ and of the outgoing wave $\beta(x,t)$. $\alpha(x,t)$ and $\beta(x,t)$ satisfy two classical wave equations:

$$\frac{\partial \alpha}{\partial t} + c\frac{\partial \alpha}{\partial x} = 0 \text{ and} \tag{10}$$

$$\frac{\partial \beta}{\partial t} - c\frac{\partial \beta}{\partial x} = 0. \tag{11}$$

The quantities $\frac{\partial \alpha}{\partial t} + c\frac{\partial \alpha}{\partial x}$ and $\frac{\partial \beta}{\partial t} - c\frac{\partial \beta}{\partial x}$ can be seen as the time derivatives $\frac{d\alpha}{dt}$ and $\frac{d\beta}{dt}$ of α and β in (x,t) along the solutions of :

$$\frac{dx}{dt} = c \text{ and } \frac{dx}{dt} = -c, \tag{12}$$

called "characteristic curves". Since $\alpha(x,t)$ and $\beta(x,t)$ are constant along these curves, α and β are called the **Riemann invariants**.

Let us introduce the following notations :

$$\alpha_0(t) = \alpha(x=0,t) \text{ and } \beta_0(t) = \beta(x=0,t). \tag{13}$$

Then, the behavior of $\alpha(x,t)$ is a time delay system from α_0 :

$$\alpha(x,t) = \alpha_0(t - \frac{x}{c}) \tag{14}$$

and conversely we have :

$$\beta(x,t) = \beta_0(t + \frac{x}{c}). \tag{15}$$

We will denote in the sequel τ_p the delay due to the length of the pipe :

$$\tau_p = \frac{L}{c}. \tag{16}$$

3. PHYSICAL MODELS OF THE JET CHANNEL AND THE MOUTH

3.1 The case of the slide flute

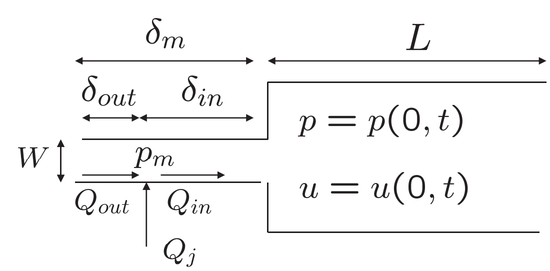

Fig. 3. The 1D model of the mouth

In [Verge, Caussé et al.(1994)], the two-dimensional geometry of the mouth is modeled in a low frequency plane wave approximation by a one-dimensional representation, by an equivalent pipe segment of length δ_m (see Fig. 3) taking into account the constriction of the pipe at the blowing end.

In this one-dimensional representation, the flue exit, when the jet is formed, is located at an acoustic distance δ_{out} from the outside and δ_{in} from the entrance of the resonator.

At the flue exit, because the region is compact, one can apply the mass conservation law :

$$Q_j + Q_{out} = Q_{in} \tag{17}$$

where Q_j, Q_{out} and Q_{in} are respectively the jet flow, the flow in the portion δ_{out} and δ_{in} respectively, expressed in m^3/s.

The pressure p_m in the mouth at the flue exit can be related to the flow Q_{out} by the radiation impedance, which leads in the time domain to the following linear differential equation :

$$\begin{cases} p_m = c_2 \ddot{Q}_{out} - c_3 \dot{Q}_{out} \\ c_2 = \dfrac{\rho_0 r_m^2}{4cS_m} \text{ and } c_3 = \dfrac{\rho_0 \delta_{out}}{S_m} \end{cases} \tag{18}$$

where S_m is the mouth cross section at the flue exit, and r_m is the radius of a circle having the same mouth cross section, i.e. such that $\pi r_m^2 = S_m$.

Neglecting friction, the jet at the flue exit is governed by the Bernoulli equation :

$$\rho_0 l_c \frac{dU_j}{dt} + \frac{1}{2}\rho_0 U_j^2 = p_f - p_m \tag{19}$$

where U_j denotes the jet velocity in the flue channel, l_c the length of the channel, p_f denotes the excitation pressure at the entrance of the channel, generated by the mouth of the musician e.g. and p_m denotes the pressure in the mouth (see Fig. 1).

Noticing that the flow continuity is assumed at the entrance of the resonator, that is : $Q_{in}(0) = u(x = 0, t) = u_0(t)$, the pressure $p(x = 0, t) = p_0(t)$ can also be related to the pressure p_m through momentum conservation:

$$\begin{cases} p_m - p_0 = c_1 \dot{u}_0 - \Delta p \\ c_1 = \dfrac{\rho \delta_{in}}{S_m} \end{cases} \tag{20}$$

where Δp represents the pressure jump across the pipe of length δ_{in}. This pressure jump, responsible of the sound production, can be mainly decomposed in two terms :

$$\Delta p = \Delta p_{jd} + \Delta p_a \tag{21}$$

Δp_{jd} denoting the pressure jump due to the jet drive mechanism and Δp_a the vortex shedding when the flow separates at the edge of the labium which appears to be important to describe nonlinear behavior in the transient attack.

The term Δp_{jd}

As explained for example in [Verge, Caussé et al.(1994)], the pressure due to the jet-drive is determined by the time derivative of the flow source corresponding to the portion of the jet flow entering the pipe at the labium. The correlation between the flue geometry and the resulting jet velocity profile has been investigated experimentally. Assuming that the jet has a Bickley profile, analytic expression can be derived which will be omitted here (see also [d'Andréa-N et al.(2008)] for computation details). It is however important to notice that this term depends on the jet position η in the mouth obtained from recent works e.g. [de la Cuadra(2005)], denoting h the jet height:

$$\eta(t) = 2\frac{u_0(t - \tau_l)h}{\pi S_m U_j} e^{\mu W} \tag{22}$$

where μ denotes the spatial amplification of the jet, W the distance between the flue exit and the labium and the delay τ_l is given by :

$$\tau_l = \frac{W}{0.3U_j}\,, \tag{23}$$

since the convection velocity has been estimated to be about the third of U_j.

We can see that the delay in the labium τ_l is time varying since it depends on U_j. But, in the transient regime of the jet velocity, U_j takes values near the origin, so that in numerical simulations, we have to wait $U_j \geq \epsilon$ for a small positive value ϵ to consider equation (23). Before that, we take $\eta = 0$. After this transient period, if U_0 denotes the asymptotic value of the jet velocity, we easily see that it depends on the excitation pressure p_f (see equation (19)), i.e.:

$$U_0 = \sqrt{2p_f/\rho_0} \tag{24}$$

and then τ_l can be considered as constant :

$$\tau_l = \frac{W}{0.3U_0}. \tag{25}$$

The term Δp_a

Using e.g. [Verge, Caussé et al.(1994)], one can express the vortex shedding term induced at the labium by the transverse acoustic flow of the pipe by the following expression :

$$\Delta p_a = -\frac{1}{2}\rho_0 \left(\frac{u_0}{\bar{\alpha}_v S_m}\right)^2 sign(u_0) \tag{26}$$

where $\bar{\alpha}_v$ is the vena-contracta factor of the flow. It can be seen that this term is dissipative, corresponding to the kinetic energy dissipation of the jet by turbulence.

3.2 The case of the brass instrument

In Fig.2, the lips of the musician are represented by a valve composed of a moving trapezoid-parallelepiped solid $\mathcal{S}$ with mass m, subjected to pressure forces F_{side} and F_{bot}, a damping $\bar{a}$, and a spring with stiffness k. We consider a one degree of freedom lip model, with the parallelepiped moving only in the vertical direction.The bottom of the mass is located by the variable $\xi(t)$ and an opened valve corresponds to $\xi > 0$. The equilibrium position at rest is denoted $\xi(t) = \xi_e$ which is supposed to be positive.

The dynamics of the solid $\mathcal{S}$ is governed by

$$m\ddot{\xi} + \bar{a}\dot{\xi} + k(\xi - \xi_e) = F_{side} + F_{bot}. \tag{27}$$

The force F_{side} due to the pressure on the sides of $\mathcal{S}$ is

$$F_{side} = (A_{side}\,\sin\theta)\,(p_m - p_0), \tag{28}$$

denoting p_m the mouth pressure and p_0 the pressure at the entrance of the resonator ($x = 0$). The area A_{side} of the lateral sides of $\mathcal{S}$ and the angle θ in Fig.2 are supposed to be constant. The force F_{bot} applied on the bottom side of $\mathcal{S}$ depends on the sign of ξ:

- If the valve is opened, F_{bot} is due to the jet pressure p_{jet} so that
$$\text{if } \xi > 0, \qquad F_{bot} = A_{bot}\,p_{jet}, \tag{29}$$
- If the valve is closed, F_{bot} is a contact force for which an empirical model can be found in [Vergez(2000)].

The equilibrium position ξ_e is a constant parameter controlled by the musician such that (27) is satisfied ($\xi = \xi_e$) with $p_m = p_{jet} = p_0 = 0$ and (arbitrary) constants m, $\bar{a}$, k.

Remark *1.* Moreover, the mouth pressure p_m is characteristic of the musician and takes the role of p_f in the mouth model for the flute.

Aperture geometry and jet

If $\xi \le 0$, the valve is closed and there is no jet. If $\xi > 0$, the valve is open and there is a jet under the solid $\mathcal{S}$. This is the case of interest that will be considered in the present paper, the case of a closed valve leading to simpler equations has been studied for example in [Strong(1990),Vergez(2000)].

The geometry of the aperture is supposed to be rectangular with an area

$$A(t) = \ell\,\xi(t). \tag{30}$$

much lower than that of the mouth section A_m so that $A \ll A_m$. The jet is considered to be governed by the Bernoulli equation (quasi-steady jet, discarded losses and particle speed neglected w.r.t. the jet velocity U_j)

$$p_m = \frac{1}{2}\rho U_j^2 + p_{jet},\ \text{if } \xi > 0, \tag{31}$$

$$U_j = 0,\ \text{if } \xi \le 0. \tag{32}$$

Remark *2.* From (31), $p_m \ge p_{jet}$ so that the jet is oriented from the mouth towards the pipe, that is,

$$U_j \ge 0.$$

Some extended modelings which authorize also negative v_{jet} have also been proposed but this case will be discarded in the present paper.

4. BOUNDARY CONDITIONS AND STATE-SPACE REPRESENTATIONS

4.1 The case of the slide flute

As it has been done in [d'Andréa-N, Coron.(2002)] in the case of an overhead crane with a **variable length flexible cable**, it is interesting to apply the following change of variable

$$x = L\sigma \tag{33}$$

to transform the system in a one with a fixed spatial domain for σ, i.e. $\sigma \in [0,1]$.

State-space representation

According to (33), if we denote:

$$\begin{cases} \tilde{\alpha}(\sigma,t) = \alpha(x,t) = \alpha(L(t)\sigma,t) \\ \tilde{\beta}(\sigma,t) = \beta(x,t) = \beta(L(t)\sigma,t) \end{cases} \tag{34}$$

equations (10) and (11) become:

$$\begin{cases} \dfrac{\partial\tilde{\alpha}}{\partial t}(\sigma,t) + \left(\dfrac{c - \dot{L}\sigma}{L}\right)\dfrac{\partial\tilde{\alpha}}{\partial\sigma}(\sigma,t) = 0 \\ \dfrac{\partial\tilde{\beta}}{\partial t}(\sigma,t) - \left(\dfrac{c + \dot{L}\sigma}{L}\right)\dfrac{\partial\tilde{\beta}}{\partial\sigma}(\sigma,t) = 0. \end{cases} \tag{35}$$

We still have two wave equations, but with time variable velocities depending on the control variable $\dot{L}$.

Let us now complete the pipe model (35) with the boundary conditions at $\sigma = 0$ (i.e. $x = 0$) and $\sigma = 1$ (i.e. $x = L$).

Boundary condition at the entrance of the resonator

Let us first consider the boundary condition at the entrance of the resonator. It can be obtained replacing p_m from equation (20) in equation (19), which leads to:

$$p_0(t) = p_f - \rho_0 l_c \frac{dU_j}{dt} - \frac{1}{2}\rho_0 U_j^2 - c_1\dot{u}_0(t) - \Delta p. \tag{36}$$

This boundary condition can be rewritten in the α and β variables using (9) and in the $\tilde{\alpha}$ and $\tilde{\beta}$ variables, using (34) which gives finally:

$$\begin{aligned} &\tilde{\alpha}(0,t) = \tilde{\beta}(0,t) + \\ &\frac{2S_p}{\rho_0 c}\left[p_f - \rho_0 l_c\frac{dU_j}{dt} - \frac{1}{2}\rho_0 U_j^2 - \frac{c_1}{2}(\dot{\tilde{\alpha}} + \dot{\tilde{\beta}})(0,t) - \Delta p\right] \end{aligned} \tag{37}$$

Remark *3.* In [d'Andréa-N et al.(2006)], the boundary condition which was previously used was the very simple one $p(x = 0, t) = 0$, i.e. $\tilde{\alpha}(0,t) = \tilde{\beta}(0,t)$, corresponding to an ideal case. Taking into account the physical models of the jet and of the mouth leads to the more realistic above condition. It can be also noticed that $p_0(t)$ now depends on $\dot{u}_0(t)$ but using (21), (26) and (22) also on $u_0(t)$ and $\dot{u}_0(t - \tau_l)$ (see [d'Andréa-N et al.(2008)] for details).

Moreover, we need the value of U_j and its time-derivative to bring up to date the boundary condition (37). So we have to solve at each time instant, the ordinary differential equation describing the dynamical evolution of U_j. This ODE is obtained from (19) where we replace p_m by its expression (18) and using equation (17) which becomes at $x = 0$:

$$Q_{out} = Q_{in} - Q_j = u_0(t) - S_e U_j \tag{38}$$

S_e denoting the cross section of the channel at the flue exit.

Finally, the ODE giving the value of $U_j(t)$ can be written:

$$c_2 S_e \ddot{U}_j - (\rho_0 l_c + c_3 S_e)\dot{U}_j + c_3 \dot{u}_0 - c_2 \ddot{u}_0 = \frac{1}{2}\rho_0 U_j^2 - p_f. \quad (39)$$

When taking realistic numerical values of the constants involved in (39) it can be seen that $c_2 S_e \simeq 10^{-9}$, which is negligible with respect to the multiplying factor of $\dot{U}_j$. Therefore, using singular perturbation arguments, one can neglect the terms in $\ddot{U}_j$ in (39) and we can consider the following ODE which will be used to evaluate U_j and its time derivative:

$$(\rho_0 l_c + c_3 S_e)\dot{U}_j = p_f - \frac{1}{2}\rho_0 U_j^2 + c_3 \dot{u}_0 - c_2 \ddot{u}_0. \quad (40)$$

Finally, the boundary condition at the entrance of the resonator consists in the two equations (37) and (40) at $x = 0$.

Boundary condition at the end of the resonator

Considering the piston mechanism which allows the translation of the slide flute, the boundary condition at the end of the flute, can be written:

$$S_p p(L,t) + F = m\ddot{L} \quad (41)$$

F being the force exerted by the motor on the slide and m the piston mass.

In a first step, one can consider that the control variable is the piston velocity $\dot{L}$, linked to the physical control F homogeneous to $\ddot{L}$, via the integrator (or cascade) system given by (41). Then if $\dot{L}$ is known, one can then compute the physical control F to apply, using e.g. "backstepping" techniques, classical in systems control theory. One can therefore consider, without loss of generality, the following boundary condition at $x = L$:

$$u(L,t) = S_p \dot{L} \quad (42)$$

which can be rewritten in the $\tilde{\alpha}$ and $\tilde{\beta}$ variables, using (9) and (34):

$$\tilde{\alpha}(1,t) + \tilde{\beta}(1,t) = 2 S_p \dot{L}. \quad (43)$$

Finally, the slide flute is completely described using equations (35), the two boundary conditions at the entrance of the resonator (37) and (40) and the one at the end of the resonator (43).

4.2 The case of the brass instrument

In the case of a brass instrument, this is the reed which vibrates and modulates the ingoing air jet maintaining the acoustic wave in the pipe.

Boundary condition at the entrance of the resonator

The boundary condition at $x = 0$ is nothing but the continuity of pressure and flow, following remarks of Hirschberg [Hirschberg(1995)]. Then, using equations (9) and the expressions (13) for $\alpha_0(t)$ and $\beta_0(t)$, we obtain :

$$p_{jet}(t) = p(0,t) = \frac{Z_c}{2}(\alpha_0(t) - \beta_0(t)), \quad (44)$$

$$A(t)\, U_j = u(0,t) = \frac{\alpha_0(t) + \beta_0(t)}{2} \quad (45)$$

where $Z_c = \frac{\rho_0 c}{S_p}$ denotes the **characteristic impedance**.

Boundary condition at the end of the resonator

At $x = L$, we can consider the non homogeneous boundary condition given by $p(L,t) = Z_L\, u(L,t)$ with a real passive impedance $Z_L > 0$ for radiation. From (9), (14) and (15), this expression translates into

$$\beta_0(t) = -\lambda \alpha_0(t - 2\tau_p), \quad (46)$$

$$\lambda = (Z_L - Z_c)/(Z_L + Z_c) \in (-1, 1), \quad (47)$$

where, for an opened pipe, $Z_L \gg Z_c$ so that $1 > \lambda > 0$.

State-space representation

A **neutral system** [Michiels, Niculescu(2007)] is a differential delay system with general expression

$$\dot{x}(t) = f\big(x(t), x(t-\tau), \dot{x}(t-\tau), v(t)\big) \quad (48)$$

$$y(t) = g\big(x(t), x(t-\tau), \dot{x}(t-\tau), v(t)\big) \quad (49)$$

where $\tau \geq 0$ is a time delay, function f is responsible for the dynamics of the system and g defines the measured quantity. As we will show, the model of the brass instrument can be represented by such a neutral system.

Let us introduce the state x, the input v, and the output y as follows : x_1 represents the lip displacement relative to the equilibrium position ξ_e, x_2 the lip velocity, x_3 the forward propagating wave at the pipe entrance. The input vector v is made of the mouth pressure and its time derivative and the measured output y is the pressure at the output of the pipe, i.e. $p(L,t)$. Using (9), (14), (15) and (46) we deduce the following expressions :

$$x(t) := [\xi(t) - \xi_e, \dot{\xi}(t), \alpha_0(t)]^T, \quad (50)$$

$$v(t) := [p_m(t), \dot{p}_m(t)], \quad (51)$$

$$y(t) = \frac{Z_c}{2}(1 + \lambda)\, \alpha_0(t - \tau_p). \quad (52)$$

Note that y is a delayed version of the pressure measured at the entrance of the pipe, with delay τ_p.

In this paper, the instrument is considered to be at rest before $t = 0$ so that x, v, and y are zero for $t < 0$. Moreover, quantities ξ_e, m, $\bar{a}$, k, A_{side}, A_{bot}, θ introduced in section 3.2 (equation (27)) are supposed to be constant. The equations of the neutral system for the brass instrument are derived below.

Mechanics of the lips

Combining equations (27) and (46) yields the following equations, assuming as we have already explained $\xi > 0$:

$$\ddot{\xi} = -a\dot{\xi} - \omega^2(\xi - \xi_e) + b_0(\alpha_0(t) + \lambda\alpha_0(t - 2\tau_p)) + b_m p_m(t) \quad (53)$$

where coefficients are given by :

$$\begin{cases} a = \dfrac{\bar{a}}{m} \; ; \; \omega^2 = \dfrac{k}{m} \\ b_0 = \dfrac{Z_c(A_{bot} - A_{side}\sin\theta)}{2m} \; ; \; b_m = \dfrac{A_{side}\sin\theta}{m} \end{cases} \quad (54)$$

Jet and acoustics

We always consider the case of interest $\xi > 0$. Using (30) and (44-46), equations (31) rewrite as follows:

$$p_m(t) = \frac{\mu}{2}\frac{(\alpha_0 - \lambda\alpha_0(t-2\tau_p))^2}{\xi^2(t)} + \frac{Z_c}{2}(\alpha_0 + \lambda\alpha_0(t-2\tau_p)) \quad (55)$$

where $\mu = \dfrac{\rho_0}{4\ell^2}$. Then, denoting

$$D(t) = \frac{Z_c}{2} + \frac{\mu}{\xi^2}(\alpha_0(t) - \lambda\alpha_0(t-2\tau_p)), \quad (56)$$

and isolating $\alpha_0(t)$ from the time derivative of (55) yields

$$\dot\alpha_0(t) = \frac{\dot p_m(t) - \lambda Z_c\dot\alpha_0(t-2\tau_p) + \mu\frac{\dot\xi(t)\left(\alpha_0(t)-\lambda\alpha_0(t-2\tau_p)\right)^2}{\xi(t)^3}}{D(t)} + \lambda\dot\alpha_0(t-2\tau_p).$$

Then, using (55) to substitute $\left(\frac{\alpha_0(t)-\lambda\alpha_0(t-2\tau_p)}{\xi(t)}\right)^2$ in the above equation leads to :

$$\dot\alpha_0(t) = \frac{\dot p_m(t) - \lambda Z_c\dot\alpha_0(t-2\tau_p) + \frac{2\dot\xi(t)\left(p_m(t) - \frac{Z_c}{2}(\alpha_0(t)-\lambda\alpha_0(t-2\tau_p))\right)}{\xi(t)}}{D(t)} + \lambda\dot\alpha_0(t-2\tau_p). \quad (57)$$

Remark *4.* For initial condition on (p_m, α_0, ξ) satisfying (55) at $t = 0$, the trajectory of (57) satisfies also (55) for $t > 0$. Equation (57) is then weaker than (55) but will be useful to derive the expected neutral state-space representation.

Finally, we can rewrite (53) and (57) under the expected neutral differential form (48), denoting the time delay :

$$\tau = 2\tau_p \quad (58)$$

and using notations (50) :

$$\dot x = f\big(x(t), x_3(t-\tau), \dot x_3(t-\tau), v(t)\big) \quad (59)$$

with

$$\begin{aligned} f_1 &= \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} x \\ f_2 &= \begin{pmatrix} -\omega^2 & -a & b_0 \end{pmatrix} x + b_0\lambda x_3(t-\tau) + b_m v_1 \\ f_3 &= \frac{v_2 - \lambda Z_c\dot x_3(t-\tau) + \frac{2x_2\left(v_1 - \frac{Z_c}{2}(x_3(t)+\lambda x_3(t-\tau))\right)}{x_1+\xi_e}}{D(t)} \\ &\text{with } D(t) = \frac{Z_c}{2} + \frac{\mu}{x_1^2}(x_3(t) - \lambda x_3(t-\tau)) \end{aligned} \quad (60)$$

From (52), we recall that the output $y(t)$ is given by :

$$y = \frac{Z_c}{2}(1+\lambda)\,x_3(t-\tau_p) = Cx(t-\tau_p)\,,\; C = \begin{pmatrix} 0 & 0 & \frac{Z_c}{2}(1+\lambda) \end{pmatrix}. \quad (61)$$

If we want to obtain a non delayed version of the output, we can introduce the following variable :

$$\tilde x(t) = x(t-\tau_p) \quad (62)$$

so that (59) and (61) can be rewritten :

$$\begin{cases} \dot{\tilde x} = f\big(\tilde x(t), \tilde x_3(t-\tau), \dot{\tilde x}_3(t-\tau), v(t-\tau_p)\big) \\ y(t) = C\tilde x(t). \end{cases} \quad (63)$$

We can notice that the only formal difference between (59), (61) and (63) is that the input in (63) is delayed with time τ_p, which is nothing but the fact that we can observe the physical output, i.e. the pressure at the end of the pipe, only when the traveling pressure wave has reached the end of the pipe. Therefore, we will consider in the sequel that (63) constitutes the neutral differential representation of our brass instrument for which we will elaborate an asymptotic observer in paper [d'Andréa-N et al.(2009)] for $t \geq \tau_p = \dfrac{\tau}{2}$. For sake of simplicity, we will keep in (63) the notation $x(t)$ in place of $\tilde x(t)$.

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