

On least action principles for discrete quantum scales

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Abstract. We consider variational problems where the velocity depends on a scale. After recalling the fundamental principles that lead to classical and quantum mechanics, we study the dynamics obtained by replacing the velocity by some physical observable at a given scale into the expression of the Lagrangian function. Then, discrete Euler-Lagrange and Hamilton-Jacobi equations are derived for a continuous model that incorporates a real-valued discrete velocity. We also examine the paradigm for complex-valued discrete velocity, inspired by the scale relativity of Nottale. We present also rigorous definitions and preliminary results in this direction.

Keywords: quantum operators, scale relativity.

1 Some philosophical principles for Physics

In this contribution, we first introduce some general philosophical hypotheses that are also widely discussed by several authors (see *e.g.* Bitbol [1], d’Espagnat [4], Filk and von Müller [5] among others). We set three hypotheses. The two first ones are of ontological type and the third one is concerned with experiments.

(H1)-Principle of reality. It exists a reality which is independent of any observer.

(H2)-Continuous space-time. The space-time is a continuous manifold on which the movement of particles can be described by continuous trajectories.

(H3)-Measurement and scale. The measurement of a physical quantity (time, space, velocity, energy, *etc*) involves a notion of scale.

- In classical physics, hypothesis (H2) is more constrained: trajectories are supposed to be differentiable or more regular. In this case, the particle velocity is uniquely defined by $v = \frac{dq}{dt}$ which is independent of the scale. Observe that if the trajectory is not regular (continuous but nowhere differentiable) or if some general hypothesis of continuous but non-differentiable space-time is done (as in scale relativity [12]), hypothesis (H3) remains true but the previous velocity has no meaning. On the contrary, a discrete velocity associated with a given scale can still be well-defined.

- This framework leads to a first paradigm (labelled by the letter “a” in table 1) of continuous classical physics. We recall in section 2 the main point about Euler-Lagrange and Hamilton-Jacobi equations. As noticed by Gondran [7], a complexification of the Hamilton-Jacobi framework provides a natural introduction to the Schrödinger equation. This second paradigm (letter “b” in table 1) is shortly displayed in section 3. As a consequence the differentiability of the trajectories is lost and they can be interpreted in terms of Brownian motion (see *e.g.* Nelson [11]).
- In this contribution, we develop a scale point of view based on the analysis of reality associated with observations at a given discrete scale. We develop in section 4 a paradigm (labeled with the letter “c” in table 1) based on the knowledge of real-valued discrete velocities. In other words, the velocity at a given scale remains a real number. The idea of introducing discrete operators as fundamental principles of mechanics and quantum mechanics has been proposed by several authors as Greenspan [8], Friedberg and Lee [6] and recently by Khrennikov *et al.* [9, 10] as well as Otake and Sasaki [13]. Nevertheless, our approach does not follow the paradigms suggested by the above references. Our objective is to develop our understanding of the ideas of Nottale [12] who introduced a set of discrete complex velocities (see the label “d” in table 1). We propose some preliminary remarks in this direction in section 5.

	Continuous Geometry	Given Scale Geometry
Classical Physics	Ⓐ Hamilton-Jacobi	Ⓒ Real-valued velocity
Quantum Physics	Ⓑ Schrödinger	Ⓓ Complex-valued velocity

Table 1. Proposition of four paradigms.

2 Some classical results on Hamilton-Jacobi equations

In order to reduce the notations, a Lagrangian function $L(x, v)$ independent of the time is given. To fix the ideas, this Lagrangian can be chosen as

$$L(x, v) = \frac{1}{2} m v^2 - \varphi(x). \quad (1)$$

The potential energy $\varphi(x)$ structures the space-time with objects governed by physical laws (H1), whereas the kinetic energy $K(v) \equiv \frac{mv^2}{2}$ catches the dynamics through the velocity. Consider a regular trajectory $\theta \mapsto X(\theta)$ for $0 \leq \theta \leq t$ and the associated action

$$A(t, X(\bullet)) = \int_0^t L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) d\theta.$$

For an arbitrary variation δt and for all C^1 -functions X and associated variations δX , we introduce the variation δA of the action. It is given by:

$$\begin{aligned}\delta A(t, X(\bullet)) &= L\left(X(t), \frac{d}{d\theta}X(t)\right) \delta t \\ &+ \int_0^t \left[\partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \delta X(\theta) + \partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \frac{d}{d\theta} \delta X(\theta) \right] d\theta\end{aligned}$$

and after integrating by parts:

$$\begin{aligned}\delta A &= L\left(X(t), \frac{d}{d\theta}X(t)\right) \delta t + \int_0^t \frac{d}{d\theta} \left[\partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \delta X(\theta) \right] d\theta \\ &+ \int_0^t \left(\partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) - \frac{d}{d\theta} \left[\partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \right] \right) \delta X(\theta) d\theta.\end{aligned}\quad (2)$$

- Let q_0 be fixed and consider the class of functions $\mathcal{C}_{q_0}^1(0, t) = \{X \in \mathcal{C}^1(0, t) \text{ such that } X(0) = q_0\}$. Notice that the difference between two functions of $\mathcal{C}_{q_0}^1(0, t)$ belongs to $\mathcal{C}_0^1(0, t)$. Thus, if $\delta X \in \mathcal{C}_0^1(0, t)$, then $\delta X(0) = 0$. Vanishing the first variations of the action leads to the well-known Euler-Lagrange equation given by

$$\begin{aligned}\partial_x L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) - \frac{d}{d\theta} \left[\partial_v L\left(X(\theta), \frac{d}{d\theta}X(\theta)\right) \right] &= 0, \\ \text{with } X(0) &= q_0, X(t) = q.\end{aligned}\quad (3)$$

Moreover, for any arbitrary time $t > 0$ and any arbitrary state q , let $X^{\text{opt}}(\bullet; t, q)$ be the solution $X(\bullet)$ in $\mathcal{C}_{q_0}^1(0, t)$ of the Dirichlet boundary problem given by the Euler-Lagrange equation (3). Observe that $X^{\text{opt}}(\bullet; t, q)$ is parameterized by the time of arrival t and the value q , as precised in (3). At fixed time t and position q , the optimal trajectory $X^{\text{opt}}(\bullet; t, q)$ is supposed to exist and to be unique. We have the initial condition $X^{\text{opt}}(0; t, q) = q_0$ and the final condition $X^{\text{opt}}(t; t, q) = q$. Moreover the trajectory $\theta \mapsto X^{\text{opt}}(\theta; t, q)$ has a velocity at time t and position q equal to $\partial_\theta X^{\text{opt}}(\theta; t, q) \Big|_{\theta=t}$ that can also be considered as a “natural” velocity $\frac{dq}{dt}(t) = \partial_\theta X^{\text{opt}}(\theta; t, q) \Big|_{\theta=t}$.

- Let the momentum $p(t, q)$ be defined by

$$p(t, q) = \partial_v L(q, \partial_\theta X^{\text{opt}}(t; t, q)) \quad (4)$$

and the optimal action $S(t, q)$ as the action along the optimal trajectory :

$$S(t, q) = A(t, X^{\text{opt}}(\bullet; t, q)). \quad (5)$$

At fixed time t , due to the Euler-Lagrange equation (3), we deduce from (2) that $\delta A(t, X^{\text{opt}}(\bullet)) = \partial_v L(q, \partial_\theta X^{\text{opt}}(t; t, q)) = p(t, q)$. In other words, the first variation of the optimal action with respect to the final state is the momentum, namely

$$\partial_q S(t, q) = p(t, q). \quad (6)$$

If time t is varying and considering the optimal trajectory $\theta \mapsto X^{\text{opt}}(\theta; t, q)$, we have $\partial_t A(t, X^{\text{opt}}(\bullet)) = L(q, \partial_\theta X^{\text{opt}}(t; t, q))$. Writing that this quantity is

the time variation of the optimal action (5) and taking into account the velocity of the optimal trajectory at the location q , we deduce

$$\partial_t S + \partial_q S \bullet \partial_\theta X^{\text{opt}}(t; t, q) = \partial_t A(t, X^{\text{opt}}(\bullet)) = L(q, \partial_\theta X^{\text{opt}}(t; t, q)). \quad (7)$$

- Introduce now the Legendre transform of the Lagrangian L relatively to the second variable v . Suppose that the function $v \mapsto y = \partial_v L(x, v)$ is invertible and denote by $V(x, y)$ its inverse. The Hamiltonian $H(y, x)$ is classically defined by

$$H(y, x) = y \bullet V(x, y) - L(x, V(x, y)).$$

Observe that if (4) holds then $\partial_\theta X^{\text{opt}}(t; t, q) = V(q, p)$ and $H(p, q) = p \bullet V(q, p) - L(q, V(q, p))$. We deduce from (7),

$$\begin{aligned} L(q, \partial_\theta X^{\text{opt}}(t; t, q)) &= \partial_t S + (\partial_q S) \bullet \partial_\theta X^{\text{opt}}(t; t, q) \\ &= \partial_t S + p(t, q) \bullet V(q, p(t, q)). \end{aligned}$$

This leads to the well-known Hamilton-Jacobi equation

$$\partial_t S + H(\partial_q S, q) = 0. \quad (8)$$

3 How to derive the Schrödinger equation ?

The “break through” from classical Hamilton-Jacobi equations to quantum dynamics is due to Schrödinger [14]. Introduce the wave function ψ according to

$$\psi = \exp\left(i \frac{S}{\hbar}\right) \quad (9)$$

and inject this relation into (4) and (8). We get $\frac{i}{\hbar} dS = \frac{1}{\psi} d\psi$ and due to (6), we have $p = \frac{\hbar}{i} \frac{1}{\psi} \partial_q \psi$. Then Schrödinger transforms the momentum p into the so-called momentum operator P defined by $P \bullet \psi \equiv -i \hbar \partial_q \psi$. Observe that the momentum P becomes now a complex derivative operator. Starting from the usual Lagrangian, we observe that the good generalisation of quantum mechanic of v^2 is not $|v|^2$ (or PP^*) but vv (or PP in the classical formalism). Then the Hamiltonian H takes the expression $H = \frac{1}{2m} P^2 + \varphi(q) = -\frac{\hbar^2}{2m} \Delta + \varphi(q)$ and the Schrödinger equation

$$i \hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + \varphi(q) \psi \quad (10)$$

is a direct consequence of the Hamilton-Jacobi equation (8).

- An other way to derive the Schrödinger equation has been proposed by Nottale [12]. The idea consists in replacing the classical trajectory derivative $\frac{d}{dt} \equiv \partial_t + v \bullet \partial_q$ by the complex Dynkin operator $\frac{d}{dt} \equiv \partial_t + v \bullet \partial_q - i \frac{\hbar}{2m} \Delta$. Then, equation (7) takes the form: $L = \partial_t S + v \bullet (\partial_q S) - i \frac{\hbar}{2m} \Delta S$ and $\partial_t S + \frac{1}{m} (\partial_q S)^2 - i \frac{\hbar}{2m} \Delta S + \varphi(q) - \frac{m}{2} \left(\frac{1}{m} \partial_q S\right)^2 = 0$. Following Gondran [7], one can derive a complex Hamilton-Jacobi equation

$$\partial_t S + \frac{1}{2m} (\partial_q S)^2 + \varphi(q) - i \frac{\hbar}{2m} \Delta S = 0. \quad (11)$$

If we decompose the complex optimal action S into its real and imaginary parts, *id est* $S = \Sigma - i \hbar \log R$, an elementary calculus allows to transform the complex Hamilton-Jacobi equation (11) into the form proposed by Bohm and Hiley [2]:

$$\partial_t \Sigma + \frac{1}{2m} (\partial_q \Sigma)^2 + \varphi(q) - \frac{\hbar}{2m} \frac{\partial_q R}{R} = 0, \quad \partial_t R^2 + \operatorname{div} \left(\frac{R^2}{m} \partial_q \Sigma \right) = 0. \quad (12)$$

The quantum potential $Q \equiv -\frac{\hbar}{2m} \frac{\partial_q R}{R}$ is the quantity that has to be added to transform the classical Hamilton-Jacobi equation (8) into the real part of the complex Hamilton-Jacobi equation (11).

- Introduce now the change of variables (9) into the complex Hamilton-Jacobi equation (11). If we derive once again the relation $\frac{i}{\hbar} \partial_q S = \frac{1}{\psi} \partial_q \psi$ towards the space variable q , we get $\frac{i}{\hbar} \partial_q^2 S = -\frac{1}{\psi^2} (\partial_q \psi)^2 + \frac{1}{\psi} \partial_q^2 \psi$. The left hand side of the complex Hamilton-Jacobi equation (11) is now equal to

$$\frac{1}{\psi} \left[\frac{\hbar}{i} \partial_t \psi + \frac{1}{2m\psi} \left(\frac{\hbar}{i} \partial_q \psi \right)^2 + \varphi(q) \psi - i \frac{\hbar}{2m} \frac{\hbar}{i} \left(-\frac{1}{\psi} (\partial_q \psi)^2 + \partial_q^2 \psi \right) \right]$$

and the Schrödinger equation (10) is established.

4 Real-valued discrete-measured velocity at a given scale

We consider now that the classical velocity is not a relevant observable. We introduce a given strictly positive scale parameter ε , a “fat” initial condition $q_0 \in \mathcal{C}([-\varepsilon, 0])$ as a continuous function and the classical discrete so-called finite difference operators

$$(\mathrm{d}_\varepsilon^- q)(\theta) \equiv \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)), \quad (\mathrm{d}_\varepsilon^+ q)(\theta) \equiv \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)). \quad (13)$$

Let us notice that the velocity $v_\varepsilon^\pm = \mathrm{d}_\varepsilon^\pm q$ is now measured at the given scale ε by two possible schemes (13), as a consequence of the hypothesis (H3). We consider a given (final) time t strictly positive and a continuous trajectory $([-\varepsilon, t] \ni \theta \mapsto q(\theta)) \in \mathcal{C}([-\varepsilon, t])$ with the initial condition q_0 . This initial condition is not classical, q_0 is not anymore given at a time $t = 0$, but on a small interval depending on the scale ε . It has to be considered in the following sense: restricted to the interval $[-\varepsilon, 0]$, function q is equal to the given function q_0 . As in the classical case described in section 2, we introduce an action A based on a regular Lagrangian $L(x, v)$ which is similar to that introduced at the relation (1):

$$A(t, q) \equiv \int_0^t L(q(\theta), \mathrm{d}_\varepsilon^- q(\theta)) \mathrm{d}\theta. \quad (14)$$

In the following, we examine the choice of $\mathrm{d}_\varepsilon^- q$ as the observed velocity. Thus the paradigm based on this choice and (14) is studied. We have just formally replaced velocity v in the second argument of the Lagrangian (1) by the discrete velocity $v_\varepsilon = \mathrm{d}_\varepsilon^- q$. We have the following result.

• **Proposition 1. Variation of the discrete action**

The variation δA of the action A defined in (14) when trajectory q is varying by an increment δq and time by an increment δt is given by

$$\begin{aligned} \delta A &= L \delta t - \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \partial_v L(\theta + \varepsilon) \delta q(\theta) d\theta \\ &\quad + \int_{t-\varepsilon}^t \left[\partial_x L + \frac{1}{\varepsilon} \partial_v L \right](\theta) \delta q(\theta) d\theta \\ &\quad + \int_0^{t-\varepsilon} \left[\partial_x L - d_\varepsilon^+(\partial_v L) \right](\theta) \delta q(\theta) d\theta. \end{aligned} \quad (15)$$

The first integral in (15) is null *a priori* since initial condition q_0 is supposed to be fixed between $-\varepsilon$ and 0.

Proof of Proposition 1.

Since Lagrangian L in (1) is a regular function, differentiating (14) yields

$$\begin{aligned} \delta A &= L \delta t + \int_0^t (\partial_x L) \delta q(\theta) d\theta + \int_0^t (\partial_v L) \frac{1}{\varepsilon} (\delta q(\theta) - \delta q(\theta - \varepsilon)) d\theta \\ &= L \delta t + \int_0^t (\partial_x L) \delta q(\theta) d\theta + \frac{1}{\varepsilon} \int_0^t (\partial_v L) \delta q(\theta) d\theta - \frac{1}{\varepsilon} \int_{-\varepsilon}^{t-\varepsilon} (\partial_v L)(\theta + \varepsilon) \delta q(\theta) d\theta \\ &= L \delta t - \frac{1}{\varepsilon} \int_{-\varepsilon}^0 (\partial_v L)(\theta + \varepsilon) \delta q(\theta) d\theta \\ &\quad + \int_0^{t-\varepsilon} \left(\partial_x L - \frac{1}{\varepsilon} [(\partial_v L)(\theta + \varepsilon) - (\partial_v L)(\theta)] \right) \delta q(\theta) d\theta \\ &\quad + \int_{t-\varepsilon}^t \left[\partial_x L + \frac{1}{\varepsilon} \partial_v L \right](\theta) \delta q(\theta) d\theta, \end{aligned}$$

so that (15) is a consequence of the definition of the operator d_ε^+ given by (13). $\square$

• We deduce from relation (15) that an optimal trajectory satisfies the discrete version of the Euler-Lagrange equation, that is

$$\partial_x L(q(\theta), d_\varepsilon^- q) - d_\varepsilon^+ [\partial_v L(q(\theta), d_\varepsilon^- q)] = 0, \quad 0 \leq \theta \leq t - \varepsilon. \quad (16)$$

This discrete-time dynamics is formally very similar to the classical Euler-Lagrange dynamics (3). Remark that it is nothing but an implicit finite difference scheme:

$$\begin{aligned} \partial_x L \left(q(\theta), \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)) \right) - \frac{1}{\varepsilon} \partial_v L \left(q(\theta + \varepsilon), \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)) \right) \\ + \frac{1}{\varepsilon} \partial_v L \left(q(\theta), \frac{1}{\varepsilon} (q(\theta) - q(\theta - \varepsilon)) \right) = 0. \end{aligned} \quad (17)$$

From (17), it is clear that the dynamics of the optimal trajectory is that of a delay system, and more precisely,

$$q(\theta) \text{ is a function of } \theta, q(\theta - \varepsilon), q(\theta - 2\varepsilon). \quad (18)$$

Function q is the solution of the two-step finite-difference scheme (17). Because $q_0(\theta)$ is known for $-\varepsilon \leq \theta \leq 0$, the knowledge of $q(\theta)$ for $0 \leq \theta \leq \varepsilon$ is generically sufficient for solving the scheme (17) under the form (18). The knowledge

of q_0 on $[-\varepsilon, \varepsilon]$ is equivalent to the knowledge of the discrete derivative $d_\varepsilon^+ q(\theta)$ for $-\varepsilon \leq \theta \leq 0$. Let us define this initial variation $(d_\varepsilon^+ q)_0$ as

$$(d_\varepsilon^+ q)_0(\theta) = \frac{1}{\varepsilon} (q(\theta + \varepsilon) - q(\theta)), \quad -\varepsilon \leq \theta \leq 0. \quad (19)$$

- From the knowledge of $q_0(\theta)$ and $(d_\varepsilon^+ q)_0$ we construct *a priori* without major difficulty the continuous trajectory q solution of (17) of the type (18) for $0 \leq \theta \leq t$. We obtain in this way a “final state” q^f which is now a piece of trajectory q :

$$q^f(\theta) = q(t + \theta), \quad -\varepsilon \leq \theta \leq 0.$$

This leads to the functional $Q_t : (d_\varepsilon^+ q)_0 \mapsto q^f = Q_t((d_\varepsilon^+ q)_0)$ defined from $\mathcal{C}([-\varepsilon, 0])$ to $\mathcal{C}([-\varepsilon, 0])$, for q_0 fixed. We suppose this functional to be one to one. In consequence, we can suppose the optimal trajectory parameterized by the final state $q^f \in \mathcal{C}([-\varepsilon, 0])$. We denote by $S(t, q^f)$ the corresponding optimal action. We observe that at fixed q_0 , it depends only on the final time t and the final state q^f whereas the action A is a functional of all the states along the whole trajectory.

• **Proposition 2. Derivative of the optimal action**

Under a variation δq^f of the final state, the optimal action admits a variation $\delta S(t, q^f)$ given by

$$\delta S(t, q^f) \equiv \frac{\partial S}{\partial q^f} \bullet \delta q^f = \int_{t-\varepsilon}^t \left[\partial_x L + \frac{1}{\varepsilon} \partial_v L \right] (q(\theta), (d_\varepsilon^- q)(\theta)) \delta q(\theta) d\theta. \quad (20)$$

Proof of Proposition 2.

Due to the discrete Euler-Lagrange equations (16), the optimal trajectory vanishes the third term of the right hand side of the relation (15). The first one is identically null because the initial condition q_0 remains fixed. The result is then a simple consequence of the relation (15) when time t is fixed. $\square$

- In the right hand side of relation (20) the final state is not explicit. In order to exhibit the variation δq^f we introduce

$$\Gamma(t, q^f)(\theta) \equiv (\partial_v L + \varepsilon \partial_x L)(q(t + \theta), (d_\varepsilon^- q)(t + \theta)), \quad -\varepsilon \leq \theta \leq 0. \quad (21)$$

Then, $\Gamma(t, q^f) \in \mathcal{C}([-\varepsilon, 0])$ and relation (20) can be also written as

$$\frac{\partial S}{\partial q^f} \bullet \delta q^f = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma(t, q^f)(\theta) \delta q^f(\theta) d\theta. \quad (22)$$

Let us observe that expression $\Gamma(t, q^f)$ is a good candidate for a momentum variable analogous to the one that satisfies the relation (6) in differentiable mechanics.

- The natural question is now to determinate the “total variation” with time of the optimal action, *id est* the discrete analogous of the expression (7). This

is not possible if we restrict to solely continuous trajectories. Nevertheless we propose a result for a discrete variation in time of amplitude exactly equal to ε . We denote by $\tilde{q}^f$ the trajectory obtained from the final state q^f after a time extension of amplitude ε : $\tilde{q}^f(\theta) \equiv q(t + \varepsilon + \theta)$ for $-\varepsilon \leq \theta \leq 0$. Then we have a simple expression for the difference $S(t + \varepsilon, \tilde{q}^f) - S(t, q^f)$ because the two integrals in (14) operates on the same optimal trajectory:

$$S(t + \varepsilon, \tilde{q}^f) - S(t, q^f) = \int_t^{t + \varepsilon} L(q(\theta), (d_\varepsilon^- q)(\theta)) d\theta. \quad (23)$$

• **Proposition 3. Discrete variation of the optimal action**

Let ξ be a continuous function in the space $\mathcal{C}([-\varepsilon, 0])$. We have

$$S(t, q^f + \xi) - S(t, q^f) = \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[\int_0^1 \Gamma(t, q^f + \eta \xi)(\theta) d\eta \right] \xi(\theta) d\theta. \quad (24)$$

Proof of Proposition 3.

We introduce $\Phi(\eta) \equiv S(t, q^f + \eta \xi)$ for $0 \leq \eta \leq 1$. It is a derivable function of the real variable η and we have

$$\begin{aligned} \frac{d\Phi}{d\eta} &= \frac{\partial S}{\partial q^f}(t, q^f + \eta \xi) \bullet \frac{d}{d\eta}(q^f + \eta \xi) = \frac{\partial S}{\partial q^f}(t, q^f + \eta \xi) \bullet \xi \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma(t, q^f + \eta \xi)(\theta) \xi(\theta) d\theta. \end{aligned}$$

Then the relation (24) is obtained by integration relative to $\eta \in [0, 1]$ and using Fubini theorem. $\square$

Then, we present here the main result of this contribution.

• **Proposition 4. Discrete temporal variation of the optimal action**

Let $\Gamma_\varepsilon(t, q^f)$ be a mean value at final time t of the momentum introduced in (21):

$$\Gamma_\varepsilon(t, q^f)(\theta) \equiv \int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta, \quad -\varepsilon \leq \theta \leq 0. \quad (25)$$

The following discrete Hamilton-Jacobi type equation holds

$$\begin{aligned} d_\varepsilon^+ S + \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \Gamma_\varepsilon(t, q^f)(\theta) (d_\varepsilon^- q)(t + \varepsilon + \theta) d\theta \\ - \frac{1}{\varepsilon} \int_t^{t + \varepsilon} L(q(\tau), (d_\varepsilon^- q)(\tau)) d\tau = 0. \end{aligned} \quad (26)$$

Proof of Proposition 4.

We recall that $d_\varepsilon^+ S \equiv \frac{1}{\varepsilon} (S(t + \varepsilon, q^f) - S(t, q^f))$. Then we have the decomposition

$$\varepsilon d_\varepsilon^+ S = -(S(t + \varepsilon, \tilde{q}^f) - S(t + \varepsilon, q^f)) + (S(t + \varepsilon, \tilde{q}^f) - S(t, q^f)).$$

We remark also that $\tilde{q}^f(\theta) - q^f(\theta) = q(t + \varepsilon + \theta) - q(t + \theta) = \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta)$. Then we have from (24) with $\xi = \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta)$:

$$\begin{aligned} S(t + \varepsilon, \tilde{q}^f) - S(t + \varepsilon, q^f) &= \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[\int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta \right] (\tilde{q}^f(\theta) - q^f(\theta)) d\theta \\ &= \frac{1}{\varepsilon} \int_{-\varepsilon}^0 \left[\int_0^1 \Gamma(t + \varepsilon, q^f + \varepsilon \eta (d_\varepsilon^- q)(t + \varepsilon + \theta))(\theta) d\eta \right] \varepsilon (d_\varepsilon^- q)(t + \varepsilon + \theta) d\theta \end{aligned}$$

and the second term of the left hand side of the relation (26) is clear. The end of the proof is a consequence of the decomposition of $\varepsilon d_\varepsilon^+ S$ and relation (23). $\square$

- The analogy between the classical Hamilton-Jacobi equation (8) and the discrete version (26) is clear. We observe that the Lagrangian is replaced by its mean value on an interval of size ε . Moreover the natural associated momentum $\Gamma_\varepsilon(t, q^f)(\theta)$ defined at relation (25) is not *a priori* strictly equal to the momentum $\Gamma(t, q^f)(\theta)$ introduced at relation (21). This splitting at the discrete scale of the moment p satisfying both relations (4) and (6) is a real difficulty that we will consider in a future contribution.

5 Towards complex-valued discrete-measured velocity

The discrete scaled velocity $v_\varepsilon = d_\varepsilon^- q$ introduced in section 4 is purely real. We consider now a complex discrete velocity v_ε . Following an idea proposed by Nottale [12], we introduce a discrete complex derivation operator $\square_\varepsilon$ according to

$$(\square_\varepsilon q)(\theta) \equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - q(\theta - \varepsilon)) + \frac{i\mu}{2\varepsilon} (q(\theta + \varepsilon) - 2q(\theta) + q(\theta - \varepsilon)), \quad (27)$$

with $\mu^2 = 1$. We decompose the discrete operator $\square_\varepsilon q$ under the form $\square_\varepsilon q \equiv \square_\varepsilon^r q + i\mu \square_\varepsilon^i q$. We have

$$\begin{aligned} (\square_\varepsilon^r q)(\theta) &\equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - q(\theta - \varepsilon)) = \frac{1}{2} (d_\varepsilon^+ q(\theta) + d_\varepsilon^- q(\theta)) \\ (\square_\varepsilon^i q)(\theta) &\equiv \frac{1}{2\varepsilon} (q(\theta + \varepsilon) - 2q(\theta) + q(\theta - \varepsilon)) = \frac{1}{2} (d_\varepsilon^+ q(\theta) - d_\varepsilon^- q(\theta)). \end{aligned} \quad (28)$$

The real part $\square_\varepsilon^r q$ is the standard time derivative for regular trajectories when ε goes to 0. The imaginary part $\square_\varepsilon^i q$ is asymptotically null for a regular function and accounts for the slope jump at a given time. This framework has been proven to be well-posed by Cresson and Greff [3] introducing a limit when ε goes to zero in a well-defined projection functional space.

- As remarked previously, the appropriate generalization of the kinetic energy $\frac{m}{2} v^2$ is obtained by taking the (complex) square of the momentum operator. So in the expression of the Lagrangian we have to replace v^2 by $(\square_\varepsilon q)^2$. We set

$$K_\varepsilon \equiv \frac{m}{2} (\square_\varepsilon q)^2 = \frac{m}{2} [(\square_\varepsilon^r q)^2 - (\square_\varepsilon^i q)^2 + 2i\mu (\square_\varepsilon^r q) (\square_\varepsilon^i q)]. \quad (29)$$

If K_ε is real, *i.e.* $\text{Im } K_\varepsilon = 0$, the product $(\square_\varepsilon^r q)(\square_\varepsilon^i q)$ is null and two cases occur.

(i) If $K_\varepsilon \geq 0$, then $\square_\varepsilon^i q = 0$ and we have a natural reference to a regular trajectory.

(ii) If $K_\varepsilon < 0$, then $\square_\varepsilon^r q = 0$. The position $q(\theta)$ is essentially unchanged during one ε -step but the jump is not null and the direction of the trajectory has changed abruptly.

If the kinetic energy is imaginary, $\text{Re } K_\varepsilon = 0$ and we have $(\square_\varepsilon^r q)^2 = (\square_\varepsilon^i q)^2$ that implies $d_\varepsilon^+ q(\theta) = 0$ or $d_\varepsilon^- q(\theta) = 0$. The particle has not moved just before time t or just after!

- We consider now the iterate of the operator $\square_\varepsilon$ with itself. This type of algebraic formula is natural for the extension of $\frac{d}{dt}(m \frac{d}{dt})$ in the Euler-Lagrange equation. We emphasise the role of $\mu^2 = 1$ when we consider the composed operator. We have

$$\begin{cases} \text{Re}((\square_\varepsilon \circ \square_\varepsilon)q) = (1 - \mu^2)(\square_\varepsilon^r \circ \square_\varepsilon^r)q + \mu^2(\square_\varepsilon^r \circ \square_\varepsilon^i)q \\ \text{Im}((\square_\varepsilon \circ \square_\varepsilon)q) = 2\mu(\square_\varepsilon^r \circ \square_\varepsilon^i)q. \end{cases}$$

Roughly speaking the product “jump by jump” allows to recover some regularity at a smaller scale $\varepsilon/2$.

- We propose to introduce the following complex action for $q \in \mathcal{C}([-\varepsilon, t + \varepsilon])$:

$$A_\varepsilon(t, q) \equiv \int_0^t L(q(\theta), (\square_\varepsilon q)(\theta)) d\theta = \int_0^t \left[\frac{m}{2} (\square_\varepsilon q)^2 - \varphi(q(\theta)) \right] d\theta.$$

Our working plan follows the ideas presented in sections 2 and 3. In an analogous way as the one proposed in section 3, we will consider the Euler-Lagrange optimality condition, introduce the optimal trajectories, derive a Hamilton-Jacobi like equation for the optimal value of the action. Then make the change of variable (9) to transform the evolution equation (26) into a Schrödinger type equation.

References

1. M. Bitbol. *Mécanique quantique, une introduction philosophique*, Champs-Flammarion, Paris, 1997.
2. D. Bohm, B.J. Hiley. *The Undivided Universe: An Ontological Interpretation of Quantum Theory*, Routledge, New York, 1993.
3. J. Cresson, I. Greff. “Non-differentiable embedding of Lagrangian systems and partial differential equations”, *Journal of Mathematical Analysis and Applications*, vol. 384, issue 2, p. 626-646, 2011.
4. B. d’Espagnat. *Le réel voilé ; Analyse des concepts quantiques*, Fayard, Paris, 1994.
5. T. Filk, A. von Müller. “Quantum physics and consciousness: The quest for a common conceptual foundation”, *Mind and Matter*, vol. 7, p. 59-79, 2009.
6. R. Friedberg, T.D. Lee. Discrete Quantum Mechanics, *Nuclear Physics B.*, vol. 225, issue 1, p. 1-52, october 1983.

7. M. Gondran. "Complex calculus of variations and explicit solutions for complex Hamilton-Jacobi equations", *C. R. Acad. Sci. Paris*, vol. 332, Série I, p. 677-680, 2001. "Complex analytical mechanics, complex nonstandard stochastic process and quantum mechanics", *C. R. Acad. Sci. Paris*, vol. 333, Série I, p. 593-598, 2001.
8. D. Greenspan. A new explicit discrete mechanics with applications, *J. Franklin Institute*, vol. 294, p. 231-240, 1972.
9. A. Khrennikov, Ja. I. Volovich. "Discrete time dynamical models and their quantum-like context-dependent properties", *J. Modern Optics*, vol. 51, issue 6/7, p. 113-114, 2004.
10. A. Khrennikov. "Discrete time dynamics", *Contextual approach to quantum formalism*, chapter 12, Springer, Berlin-Heidelberg-New York, 2009.
11. E. Nelson. "Derivation of the Schrödinger Equation from Newtonian Mechanics", *Physical Review*, vol. 150, p. 1079-1085, 1966.
12. L. Nottale. *Fractal space-time and microphysics: towards a theory of scale relativity*, 333 p., World Scientific, 1993.
13. S. Odake, R. Sasaki. "Discrete Quantum Mechanics", *Journal of Physics A: Mathematical and Theoretical*, vol. 44, issue 35, 353001, 2011.
14. E. Schrödinger. "Quantizierung als Eigenwertproblem (Erste Mitteilung)", *Annalen der Physik*, vol. 79, p. 361-376, "Über das Verhältnis der Heisenberg Born Jordanischen Quantenmechanik zu der meinen", *Annalen der Physik*, vol. 79, p. 734-756, 1926.