ircam Centre Pompidou

# Non-Standard Multiset


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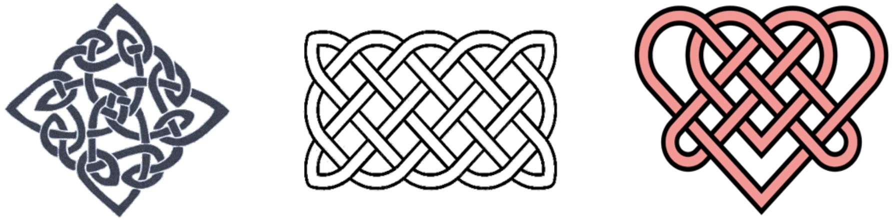

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# Gamma and beyond

- Gamma considers seriously multiset (rewriting) for programming
- However, sometimes even multisets lack of structures
- Hence:
  - Structured Gamma
  - HOCL
  - negative (abelian group) and infinite multiplicities
  - MGS
  - … ?
- Gamma is a unconventional language but based on conventional multiset. Can we parallel set theory:

  **Non well founded multisets?**

# From hydromel to hyperset (according T. Forster)

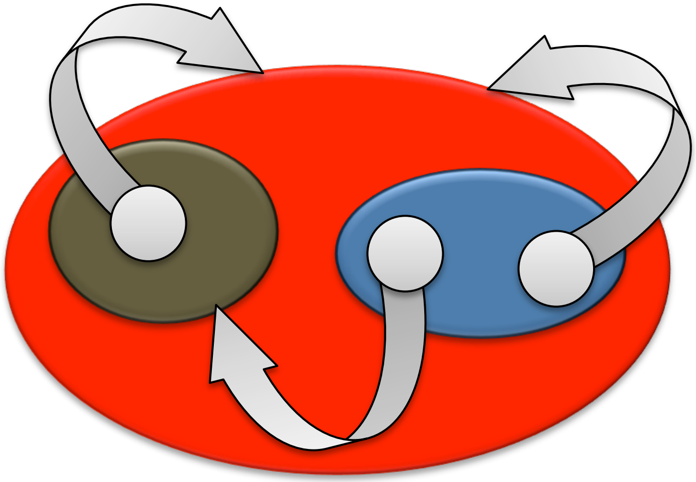
- *Hydromel* is made of *chouchen* and *chufere*
- Chouchen is made of pure hydromel
- Chufere is made of hydromel and chouchen

- ⇒ Hydromel is made of hydromel (right!)
  But how distinguishing between hydromel and chouchen?

*hydromel* = { *chouchen*, *chufere* }

*chouchen* = { *hydromel* }

*chufere* = { *hydromel*, *chouchen* }



# Hyperset (= non-well-founded set)

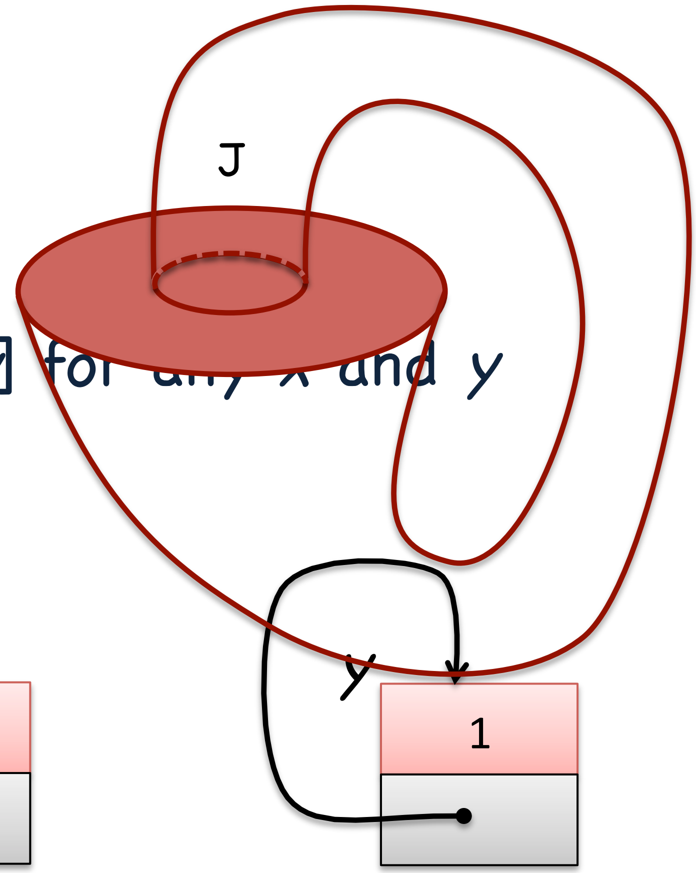
- a set *b* is a hyperset if there exists an infinite descending sequence

  $$\ldots\ a_{n+1} \in a_n \in a_{n+1} \in \ldots\ \in a_{n+1} \in b \qquad (\textit{illfounded})$$

- J = { J }

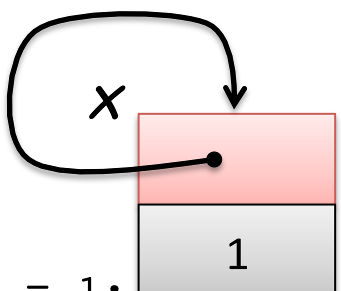
- Standard set theory (ZFC):
  every set is well-founded

- (FA) $\Rightarrow x \neq [x, y]$ and $y \neq [x, y]$ for any $x$ and $y$



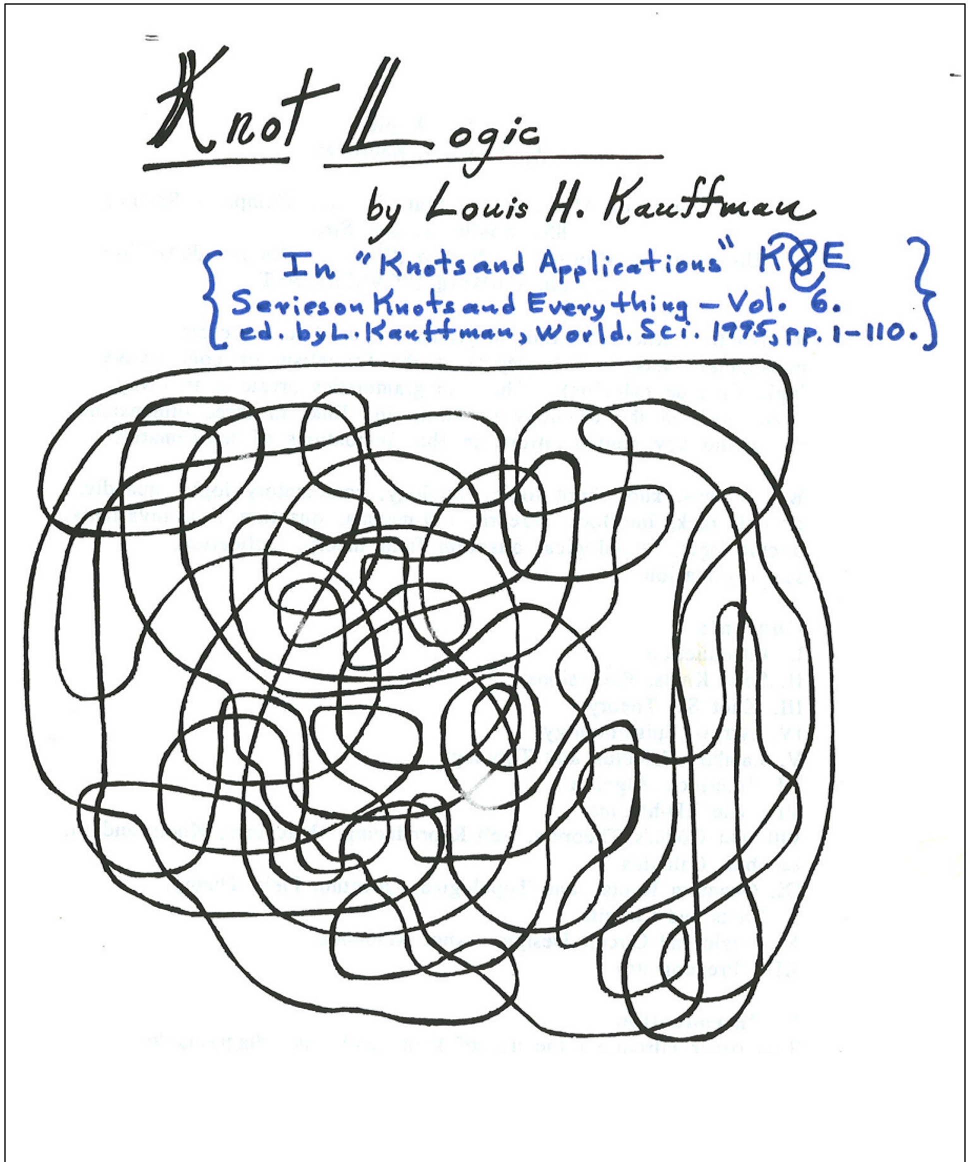

**C struct with pointers**

```
struct C = {
    C* x;
    int y;
}
C X; X.x = &c; X.y = 1;
```




```
type ('a, 'b) pair = Pair of 'a * 'b;; let rec x = Pair(x, 1);;
```

# Non Standard Multiset (following Louis Kauffman for hyperset)




- Non standard Multiset (NSM) as
  - Words
  - Planar subsets
  - Graphs
- Tools from
  - Language theory
  - Topology
  - Diagrams

to investigate NSM and check that there is no dangers

# Words on parenthesis and the nesting of sets

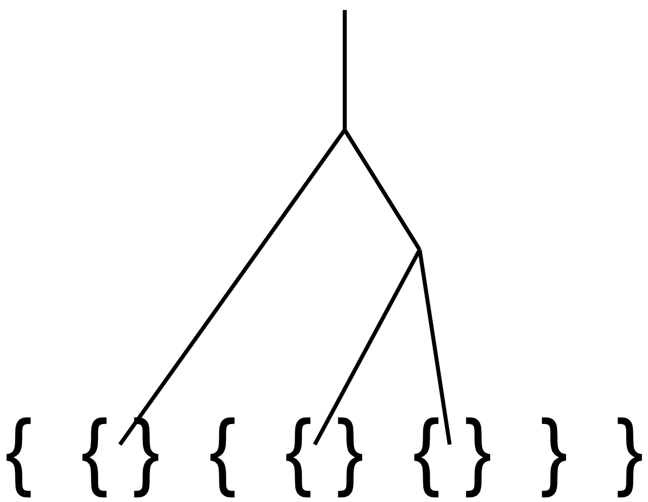
- A finite word E on { {, } } is well-formed iff
  - E is empty .
  - E = {F}G  where F and G are well-formed
- A finite ordered multiset is an expression

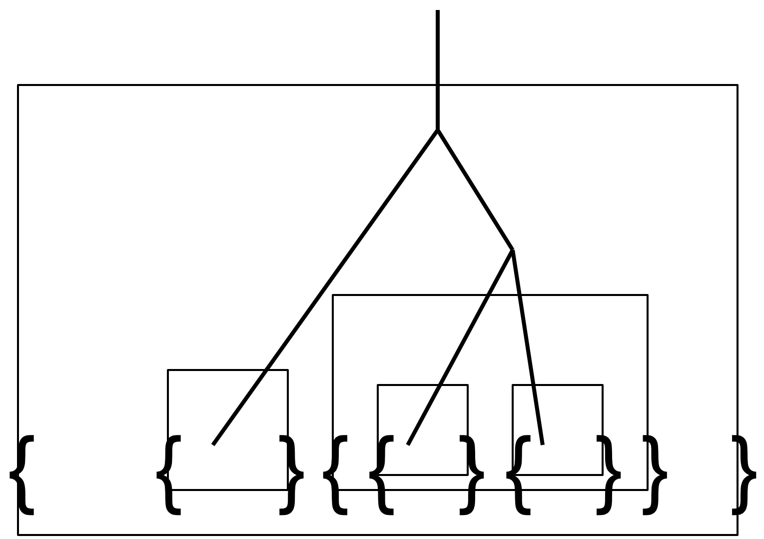
  S = { T }

  where T is well-formed
  thus T = $A_1$ $A_2$ … $A_n$  where the $A_i$ are the elements of S, are finite ordered multiset
- Finite multisets are the equivalence classes generated by XY = YX where X and Y are well formed
- Example :
  S = {{} {{}}} multiset with 2 elements {} et {{}}
  X = { {} {} {} } (three times the same element)

# Trees and boxes

- Multisets can be represented by trees
- Multisets can be represent by boxes
  (you can move and stretchs the boxes but not cross them)



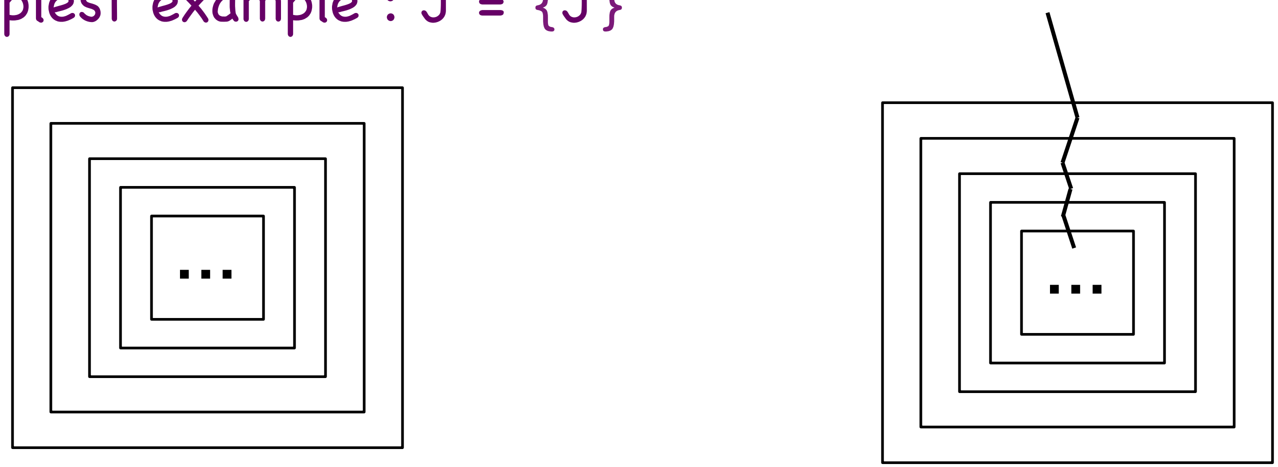



# Forms and Non Standard Multisets (NSM)

- Forms are (eventually infinite) collections of rectangles such that two rectangles are either disjoints or one included into the other

- NSMs are (eventually infinite) collections of rectangles
  - there is one outermost rectangle *R*
  - the elements inside *R* are disjoint unions of elements of NSM

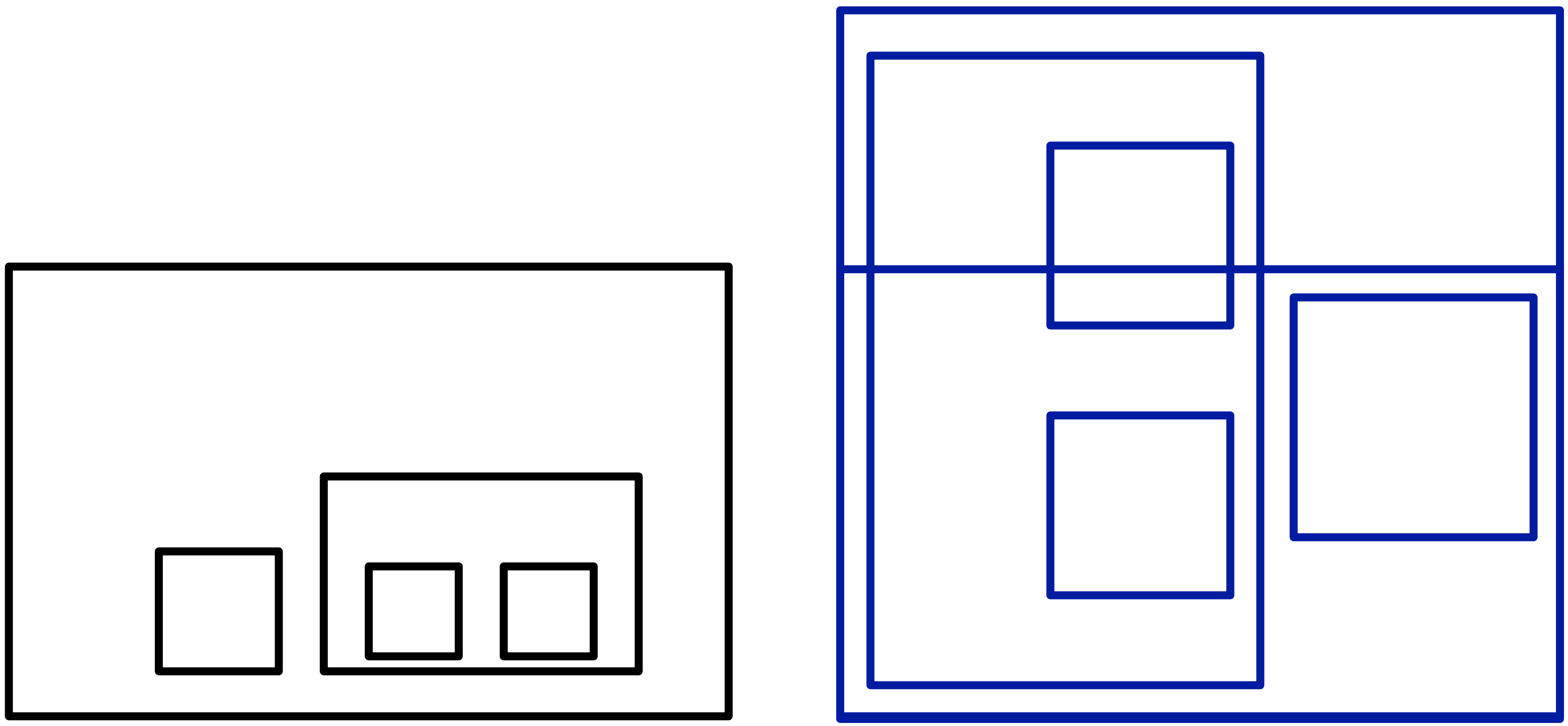
- NSM are framed forms: NSM = { Form }
- The simplest example : J = {J}



J = { { { { { { {…} } } } } } }

# NSM

- Two NSM (form) are equal if you can superpose them (= if they are homeomorph in the plane)



# NSM defined by a set of recursive equations

A = { {} B }

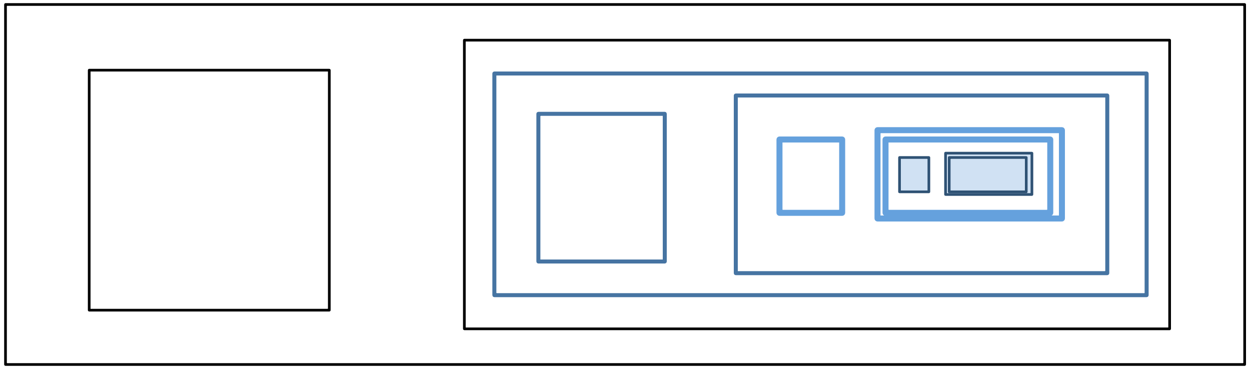
B = {A}

A = { {} B }

= { {} {A} }

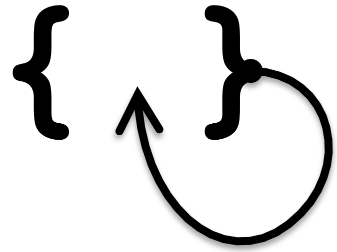
= { {} {{ {} {A} } } }

= ...

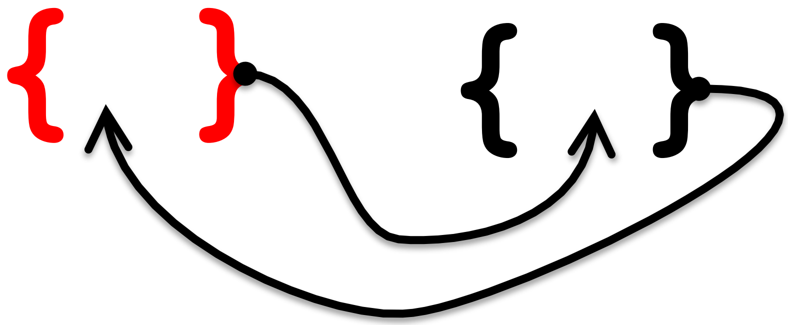


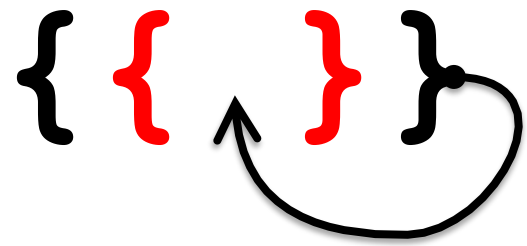
# Recursive notation

- J = {J}

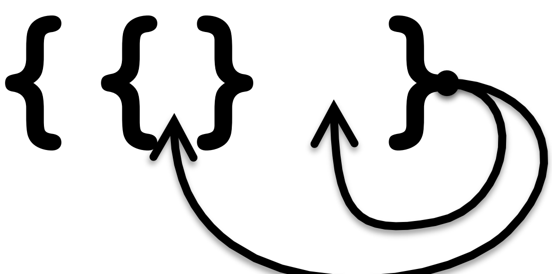


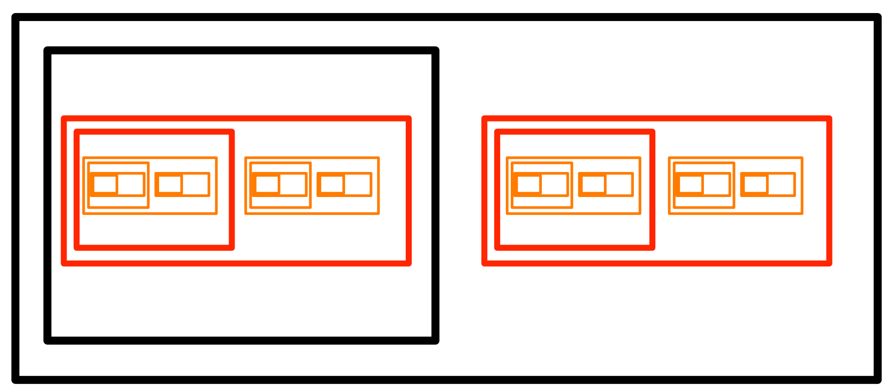
- A = {B} and B = {A} thus A = {{A}}





- F = {{F} F}





# Number of divisions of a Form

- The number $[X]_n$ of *divisions* of a form X at depth $n$

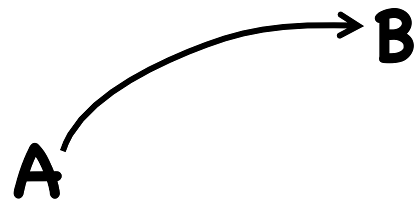
  $$[XY]_n = [X]_n + [Y]_n$$
  $$[\{X\}]_n = [X]_{n-1}$$

- For $F = \{\{F\}F\}$:

  $$[F]_n = [\{F\}]_{n-1} + [F]_{n-1}$$
  $$= [F]_{n-2} + [F]_{n-1}$$

# NSM defined by a finite set of recursive equations = directed graph (*à la* Aczel)

$B \in A$

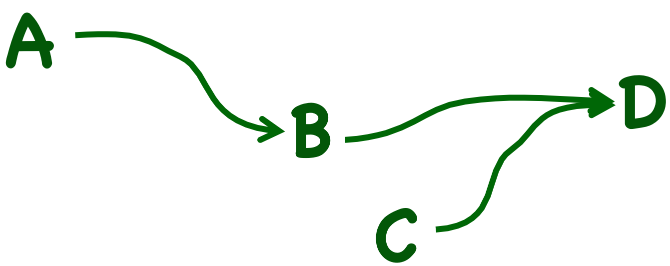


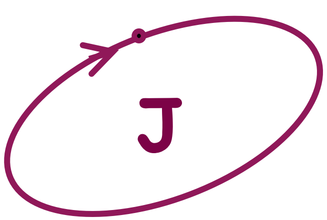

$A = \{ B \}$

$B = \{ D \}$

$C = \{ D \}$

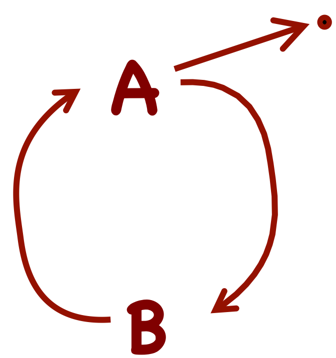
$D = \{ \ \}$



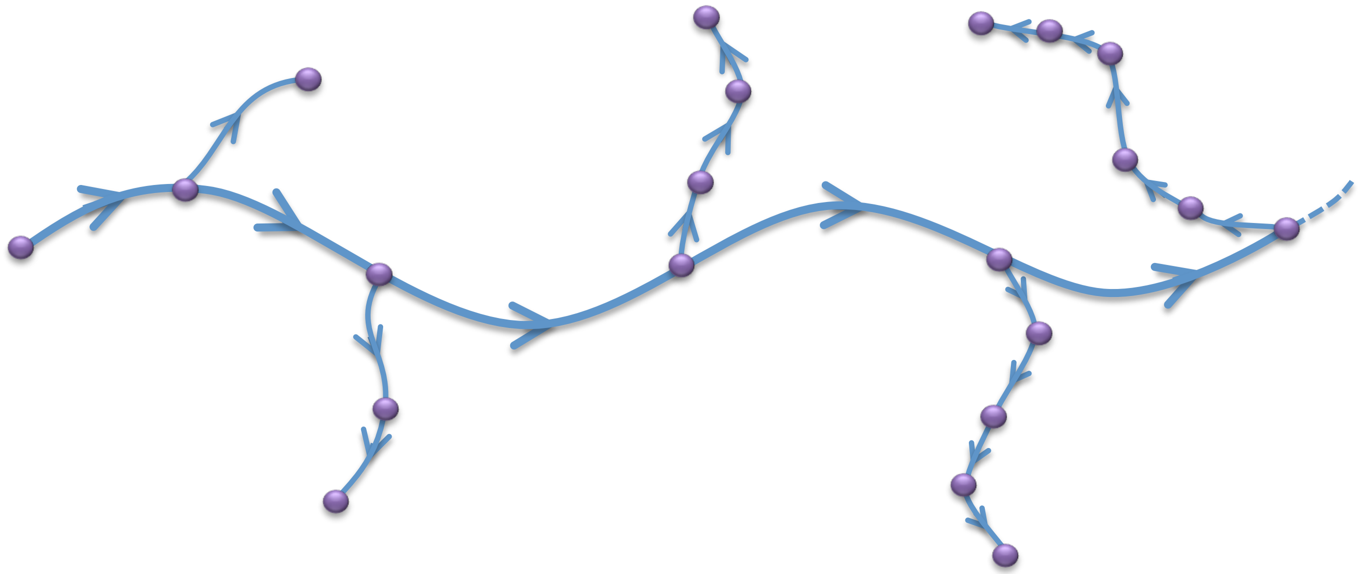



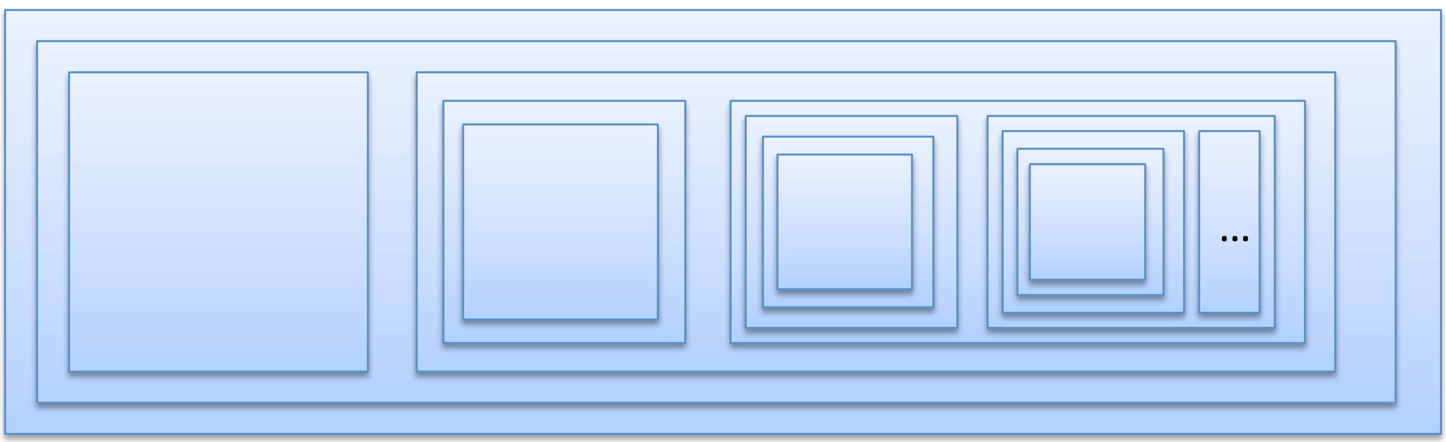

$A = \{ \{\} B \}$

$B = \{ A \}$



# They are more NSM than finite directed graphs





# Do we avoid the Russel Paradox?

- We do not refer to the set of all multisets
- An axiomatic definition of NSM will enforce hereditarily constructions. Here, this is achieved by putting in the plane already pictured NSMs.
- NSM are limits of well-founded multisets (NSS à la Aczel are less than the limits of well-founded sets)

- We can defines the Russel set of a multiset M
  $r(M) = \{ x \in M \mid x \notin x \}$
- ZFC: r(M) = M
  This is not necessarily true for NSM

| M | r(M) |
|---|---|
| J = {J} | ∅ |
| b = {1, b} | {1} |
| b = {0, {1, b}} | b = {0, {1, b}} |

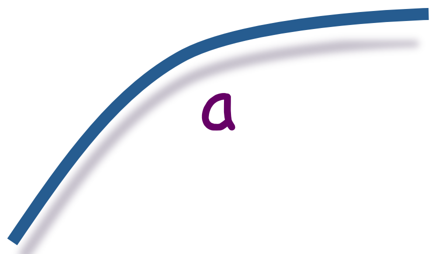
# From Form to Boolean Algebra…

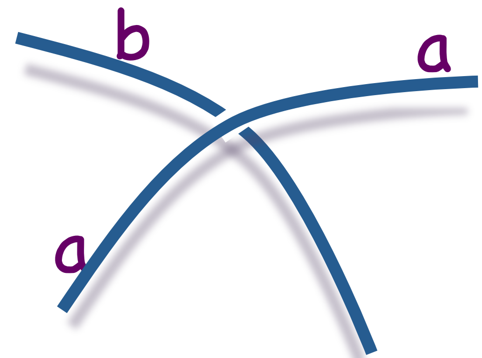
- From NSM to SET:
  add the equivalence *XX* = *X*

- From form to (almost) Boolean algebra
  - add {{}} = .
  - Example: {{}}{} = {}
  - Interpret
    - {X} as the *negation* of X
    - XY as the *disjunction* "X *or* Y"
    - {{X}{Y}} as the *conjunction* "X *and* Y"
    - {} as *true* and {{}} as *false*
  - They are more values than just *true* and *false*:
    the infinite forms
  - Infinite forms give solution to equation like
    P = not(P) that is P = {P}
    (very similar to the extension of $\mathbb{R}$ to $\mathbb{C}$ to give solution to $x^2 = -1$)

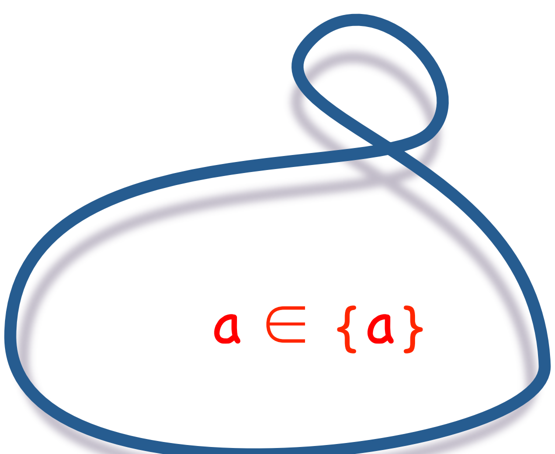
# From Form to Knot…

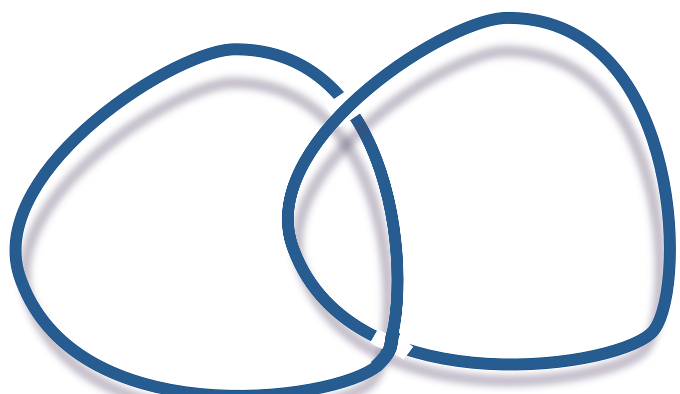
- add the equivalence *XX* = .
  (cancellation of pair instead of condensation)



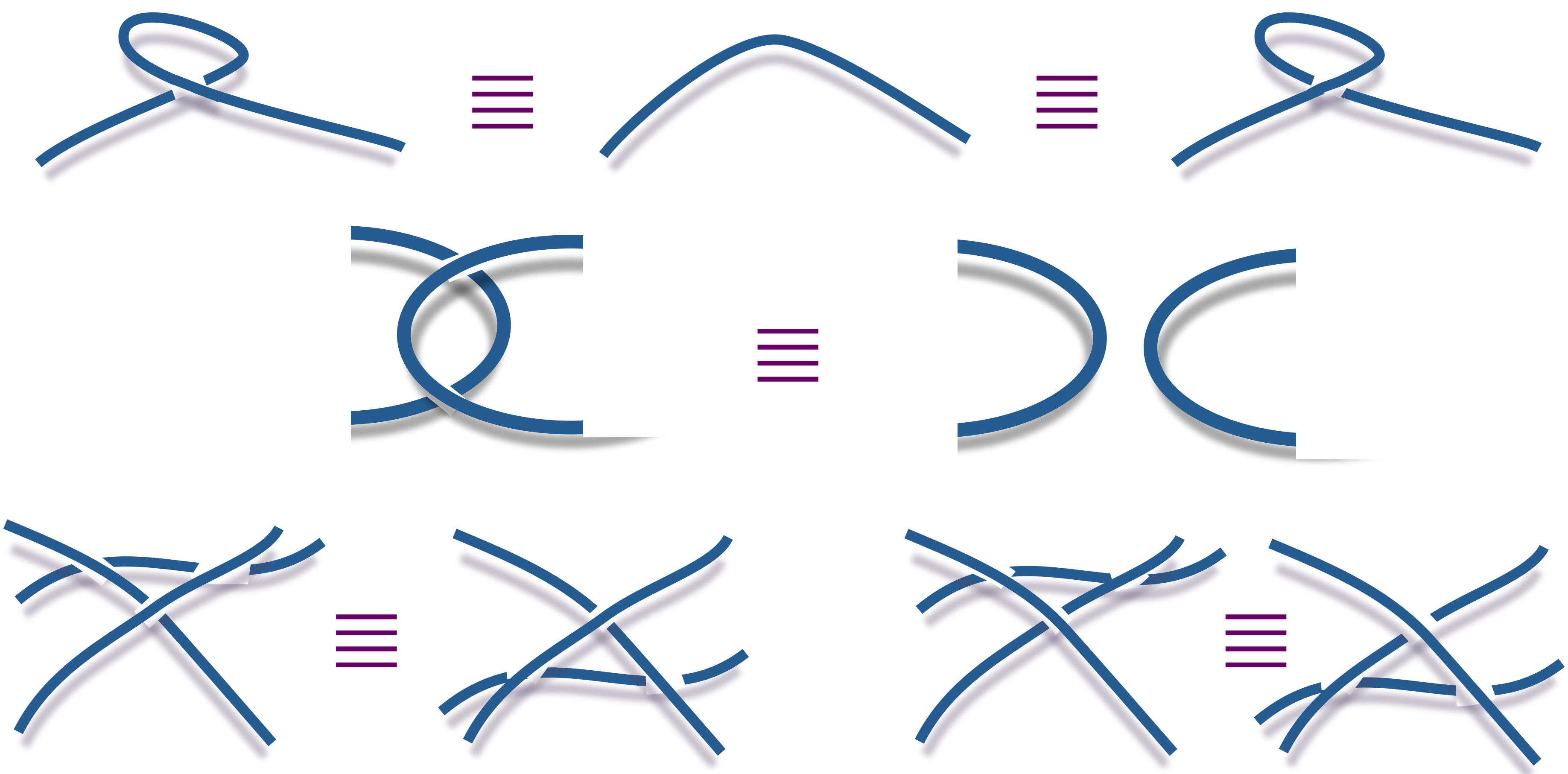

$a \in b$



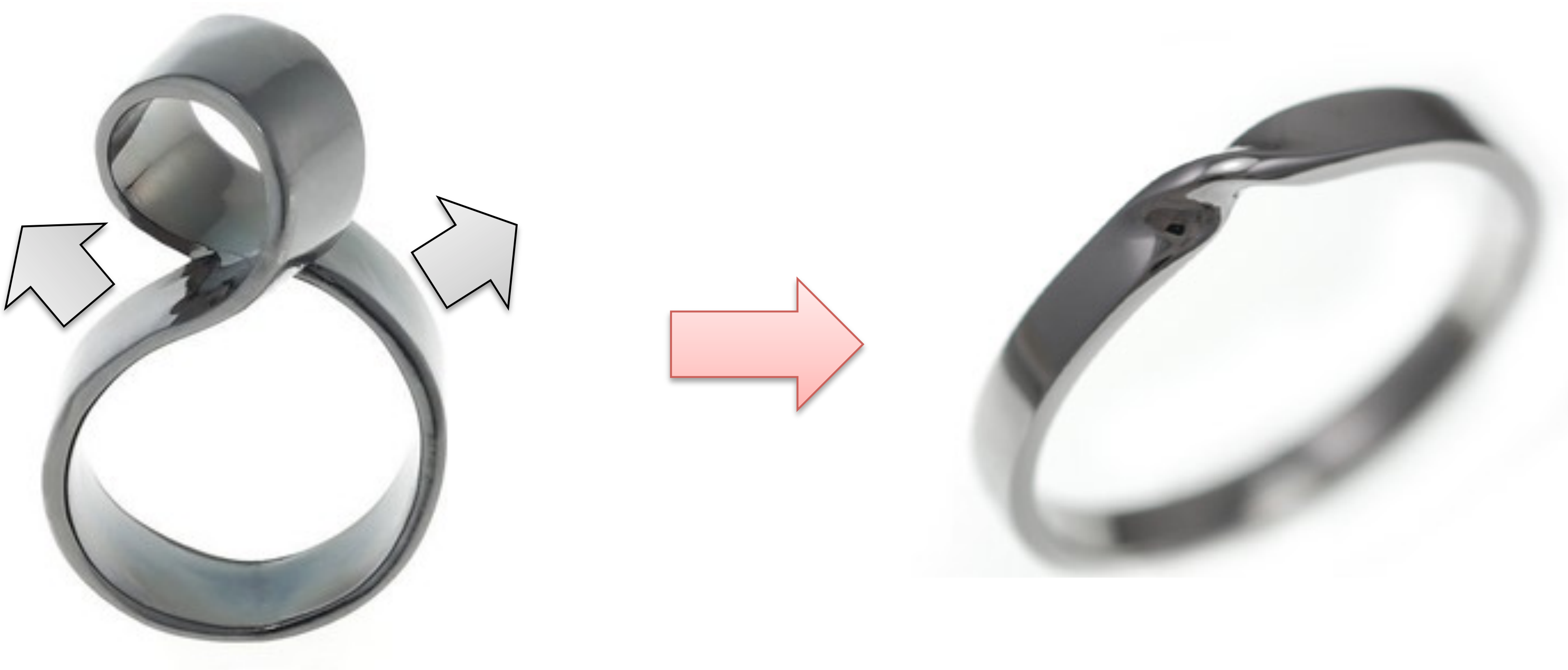
$a \in \{b\}$
$b \in \{a\}$

# Reidemeister Moves

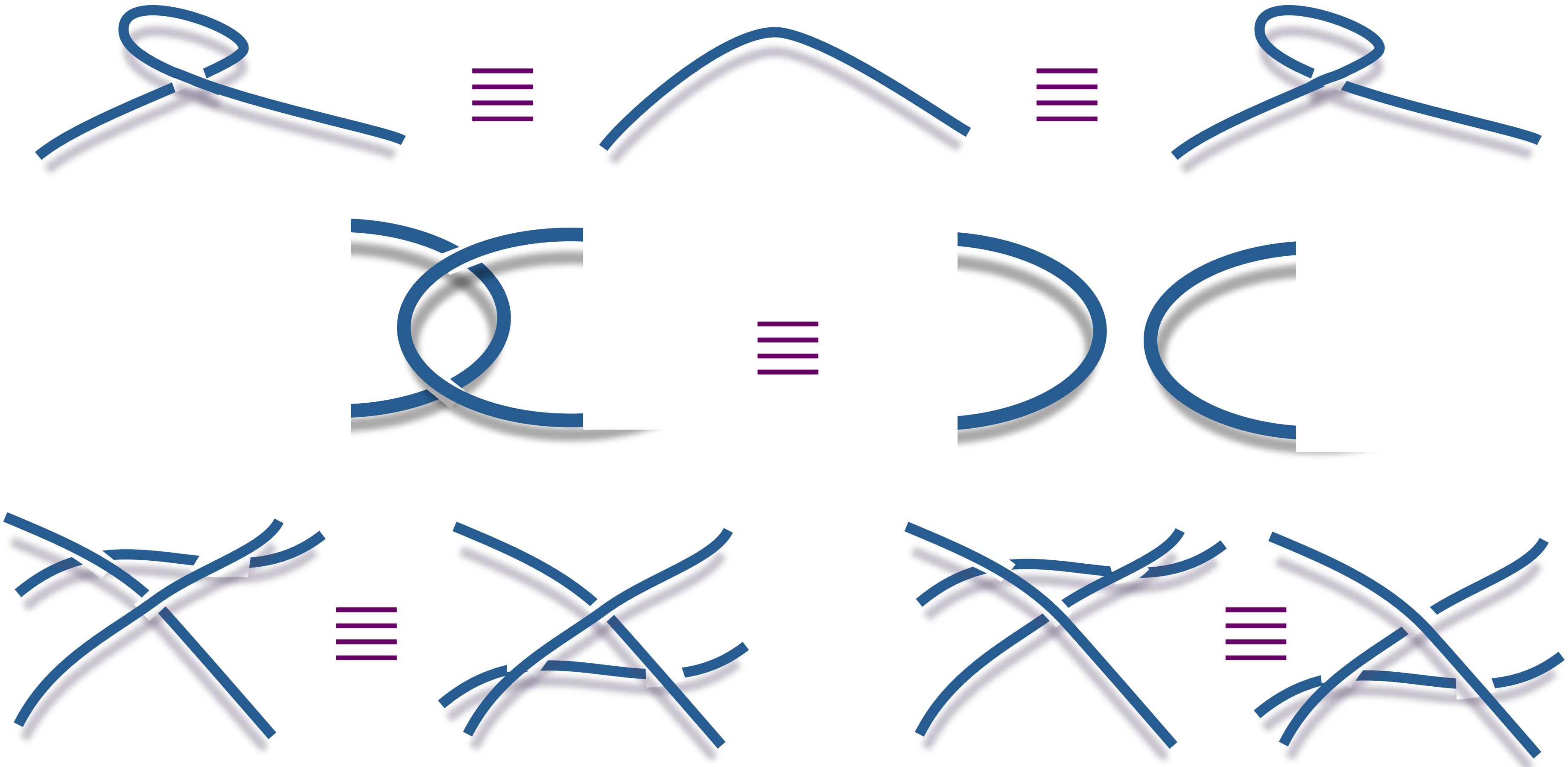


- Move 1 : enables self-membership
- Move 2 : pairs of elements disapear. So: X = {C} = {X X C} thus, look membership only in the reduced knot
- Move 3 : does not change memberships

# Ribon twist

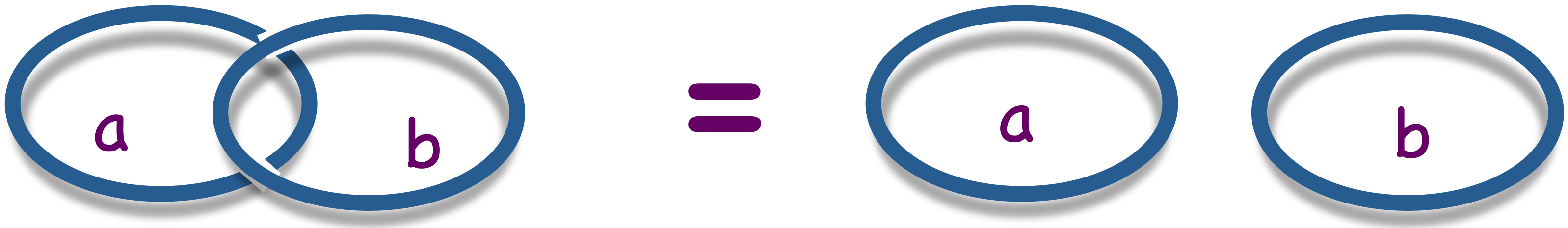


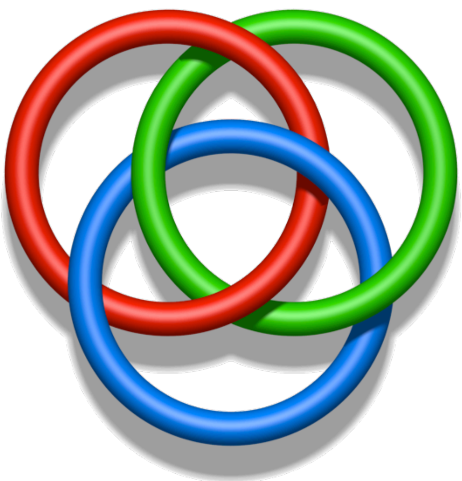
# Reidemeister Moves

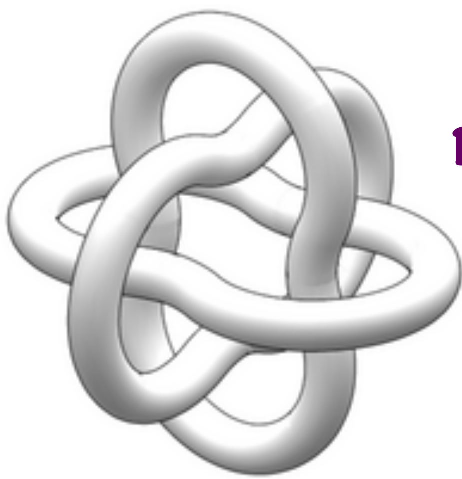


- Move 1 : enables self-membership
- Move 2 : pairs of elements disapear. So: X = {C} = {X X C}
  thus, look membership only in the reduced knot
- Move 3 : does not change memberships

# For instance…

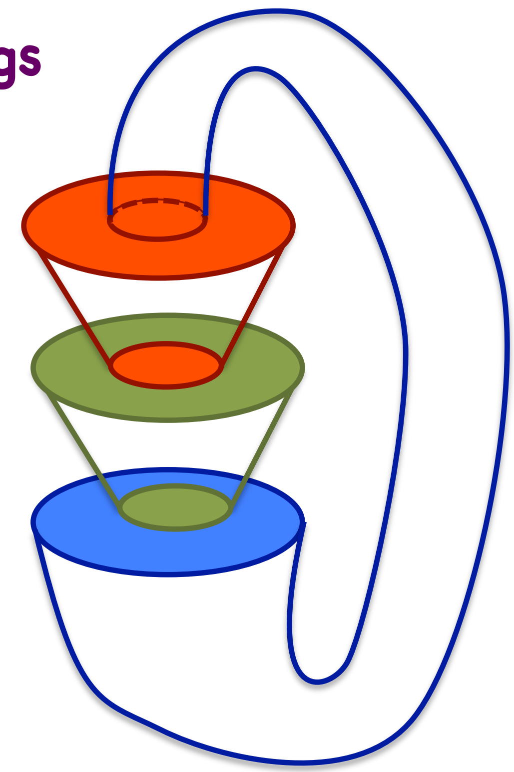




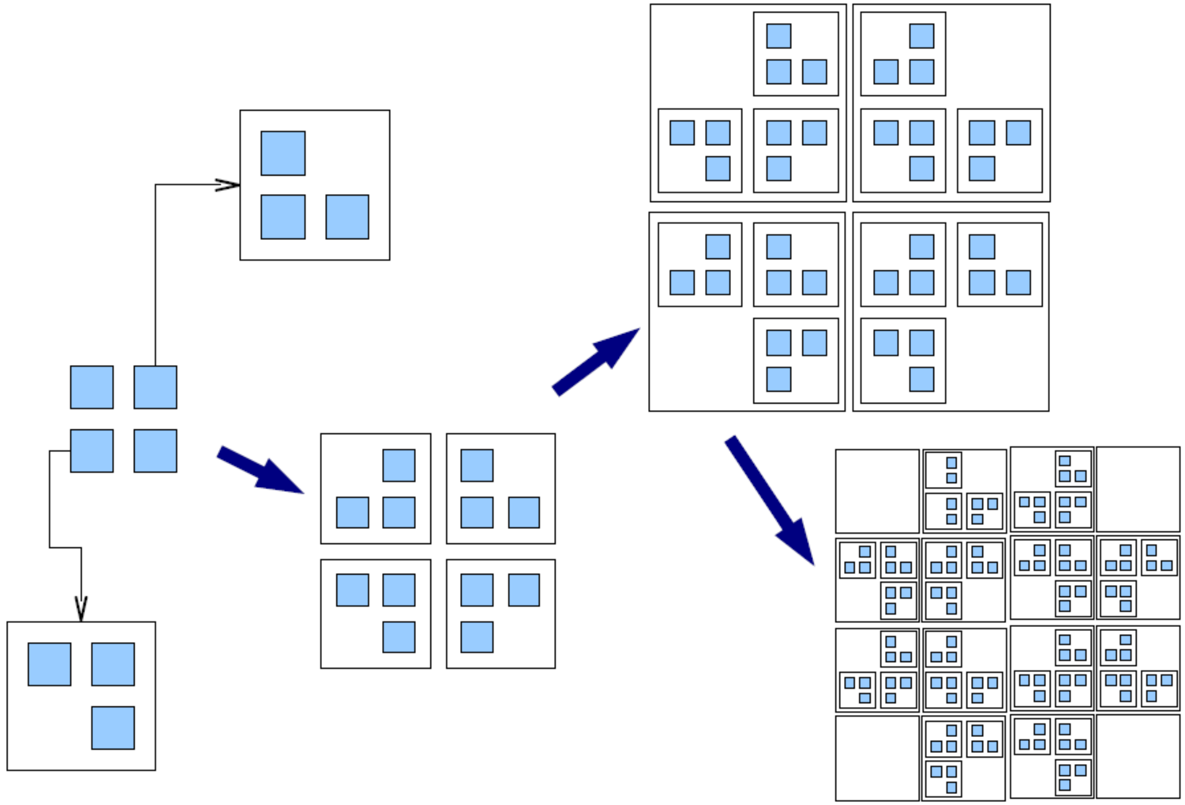
**Borromean rings**

fall apart
upon the
removal of
any one
of the triplet



| | |
|---|---|
| $a \in b$ | $a = \{c\}$ |
| $b \in c$ | $b = \{a\}$ |
| $c \in a$ | $c = \{b\}$ |

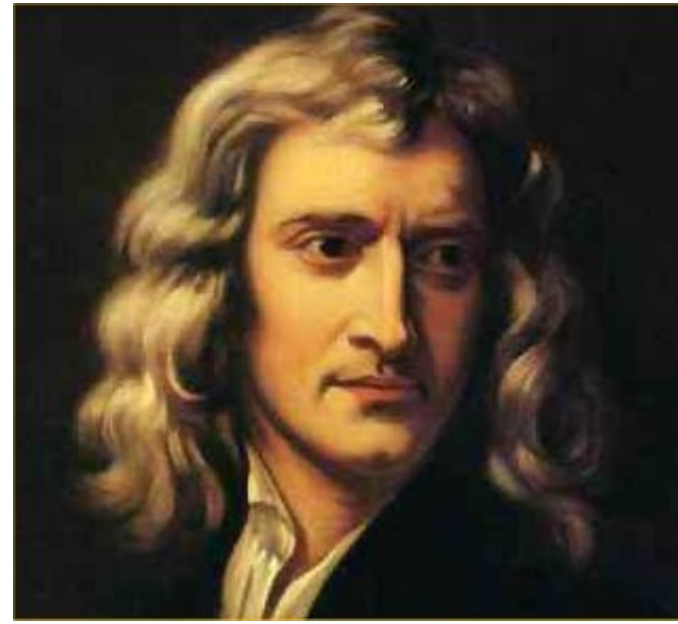
# Space, intrinsically

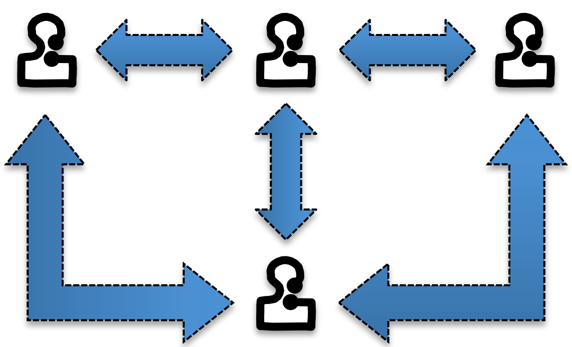


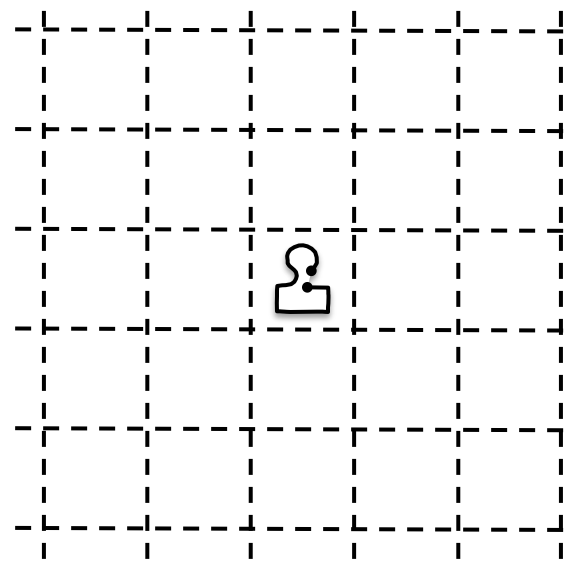
# Graphs, intrinsically

- Graph G = (V, E) avec $E \subset V \times V$
- This definition is ***extrinsic***
  V pre-exist to the graph.
  - What we want is vertices that are only the organization between them, as co-existence, not as pre-existence.
  - In addition, vertices have a position only relatively to the others vertices, not an absolute position (Lebniz *vs.* Newton)
- My motivation come from biological development where the organism build its own space

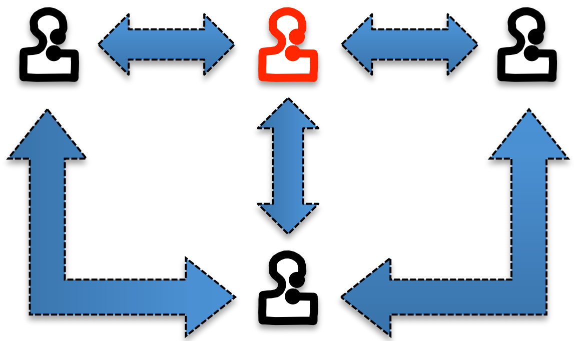
# Leibniz vs. Newton



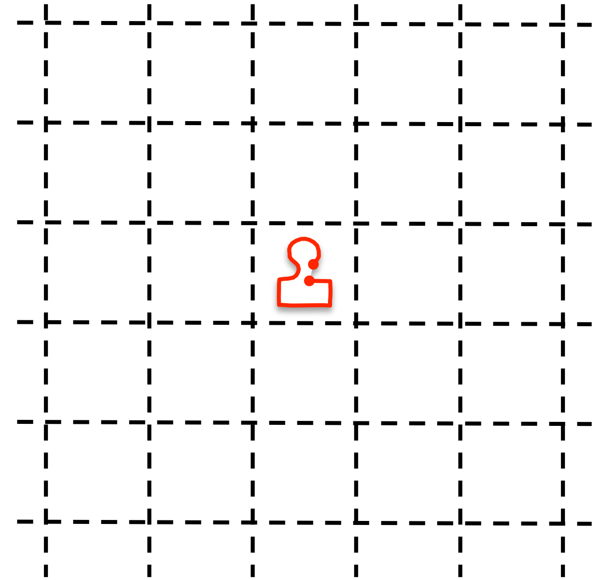


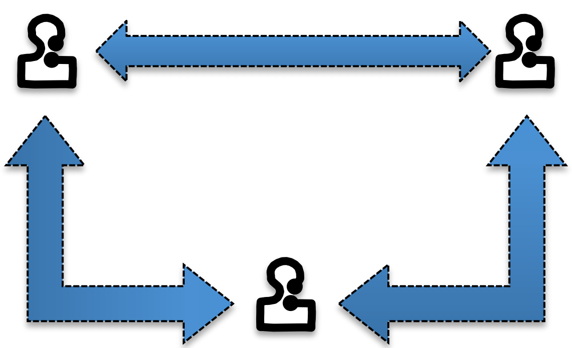


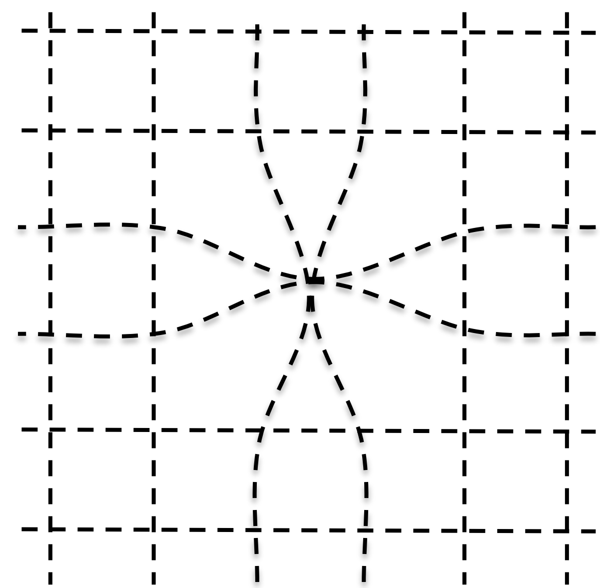
# Leibniz vs. Newton



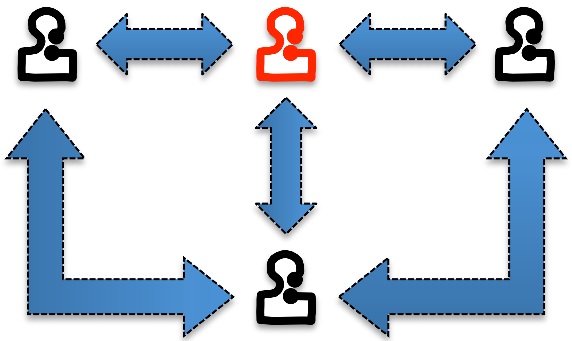
$x$ => .

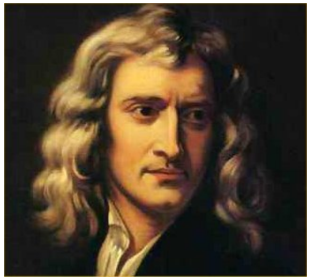




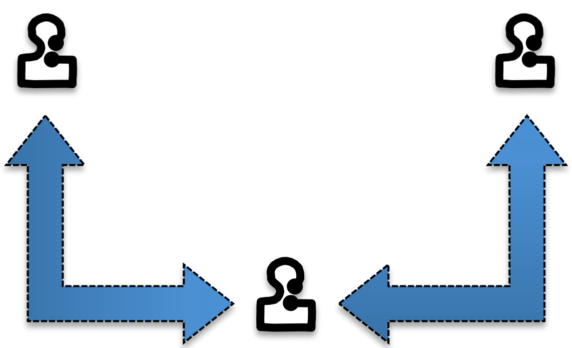


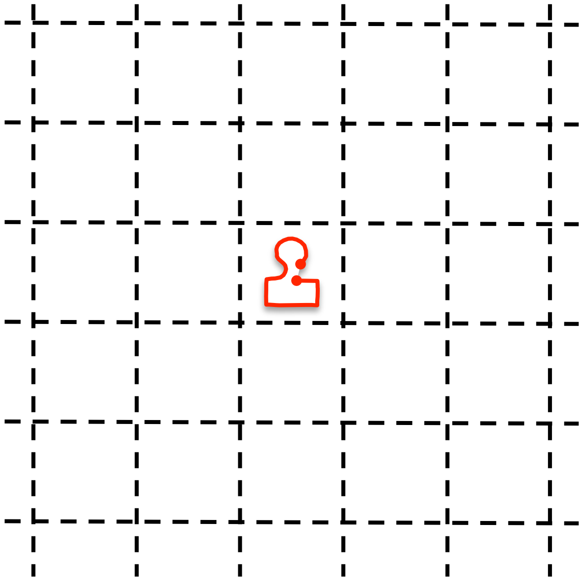
# Leibniz vs. Newton

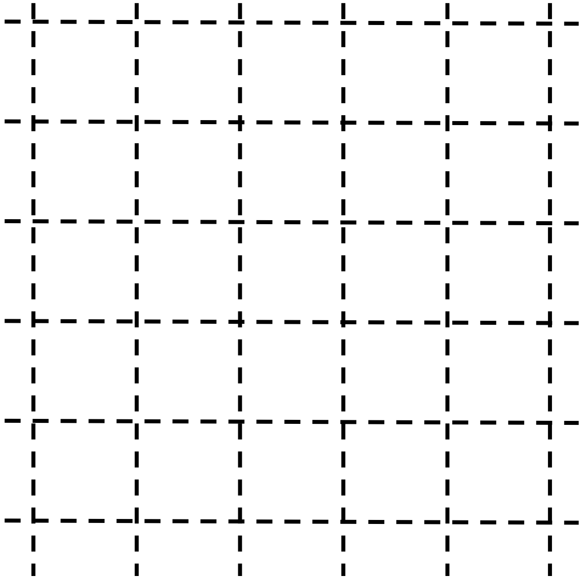




$x$ => .





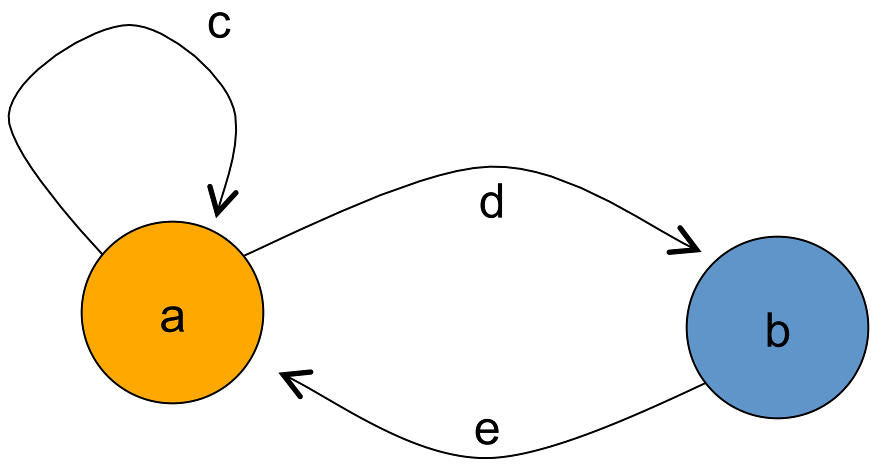


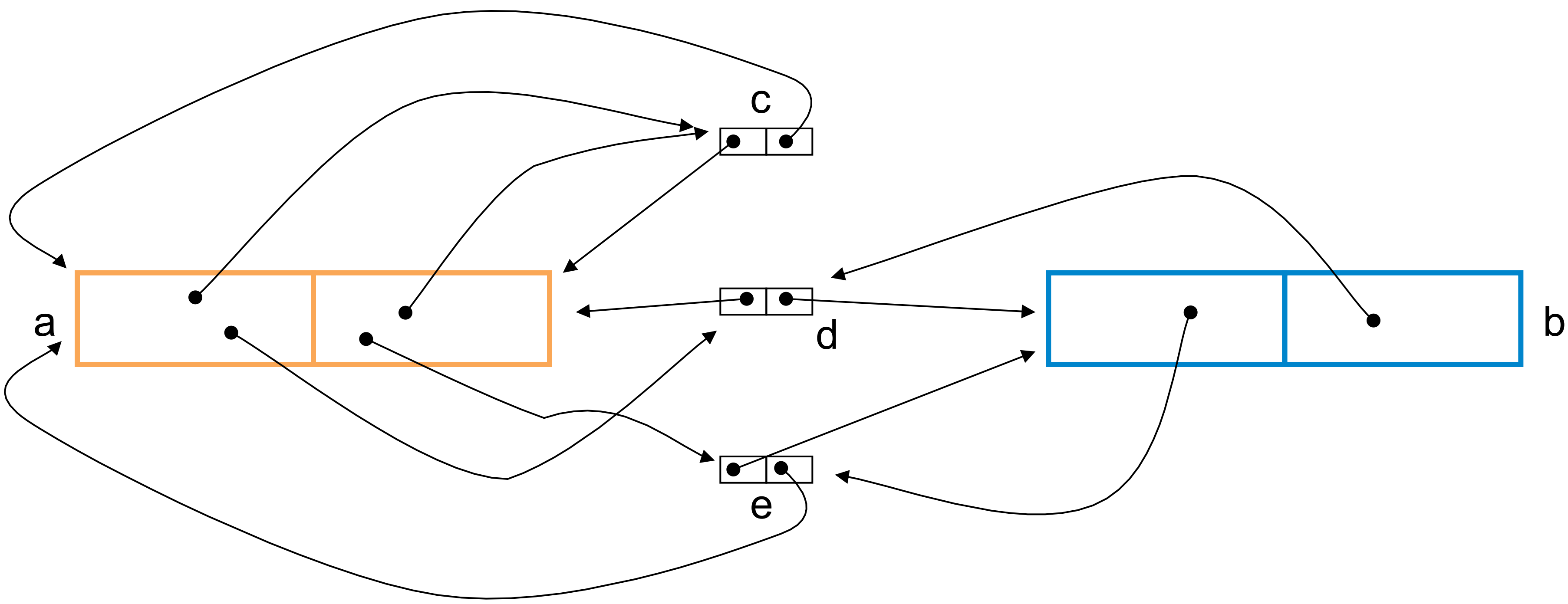
# Graph, intrinsically

A graph

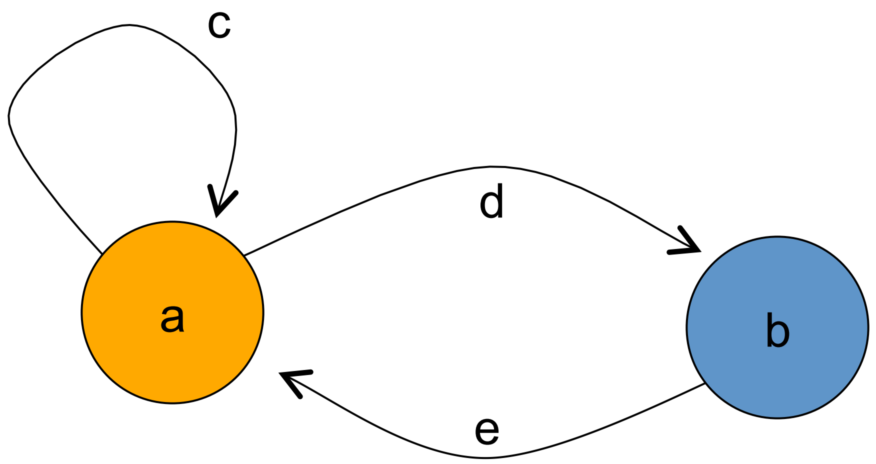
- is a pair (V, E) where V est is a set of vertices and E is a set of edges
- An edge E is a pair of vertices
- a vertex V is a pair (I,O)
  where I is the set of the ingoing edges and O is a set of the outgoing edges

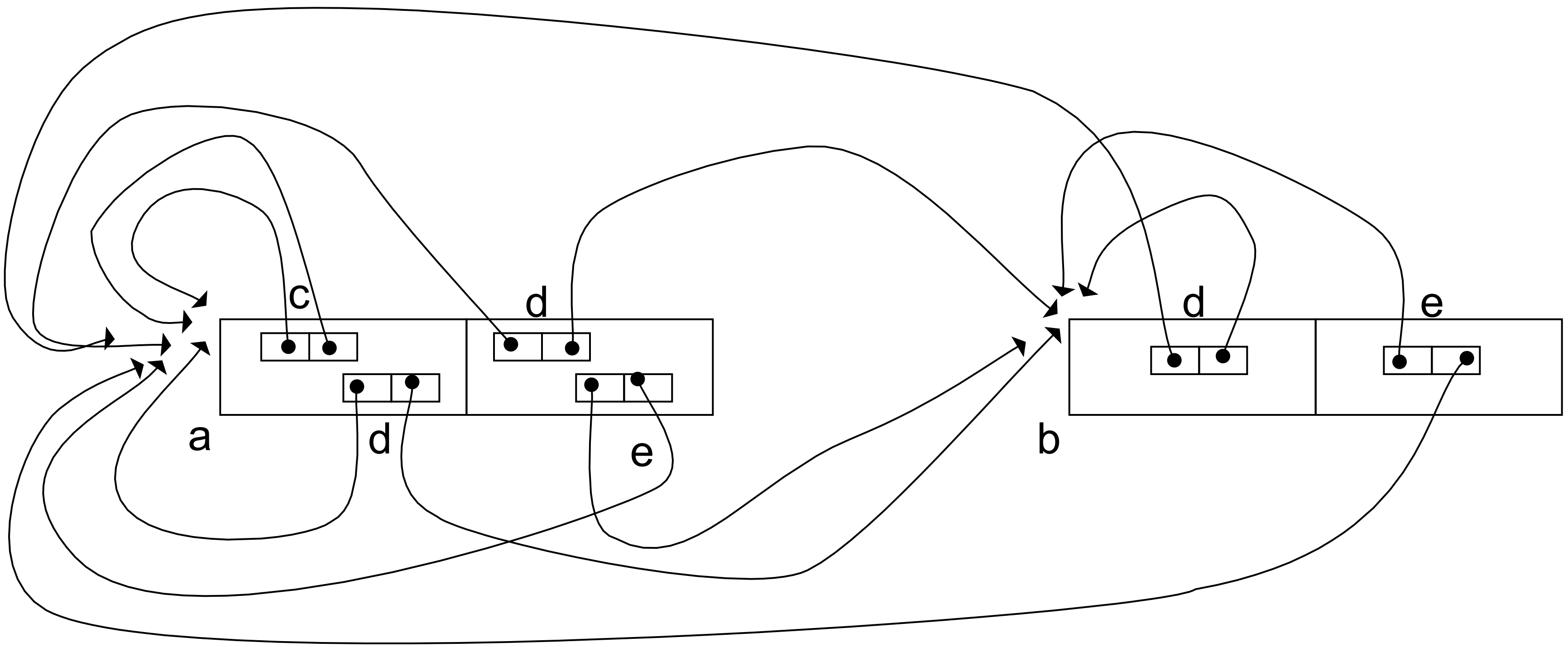
# Un graphe en soi : exemple



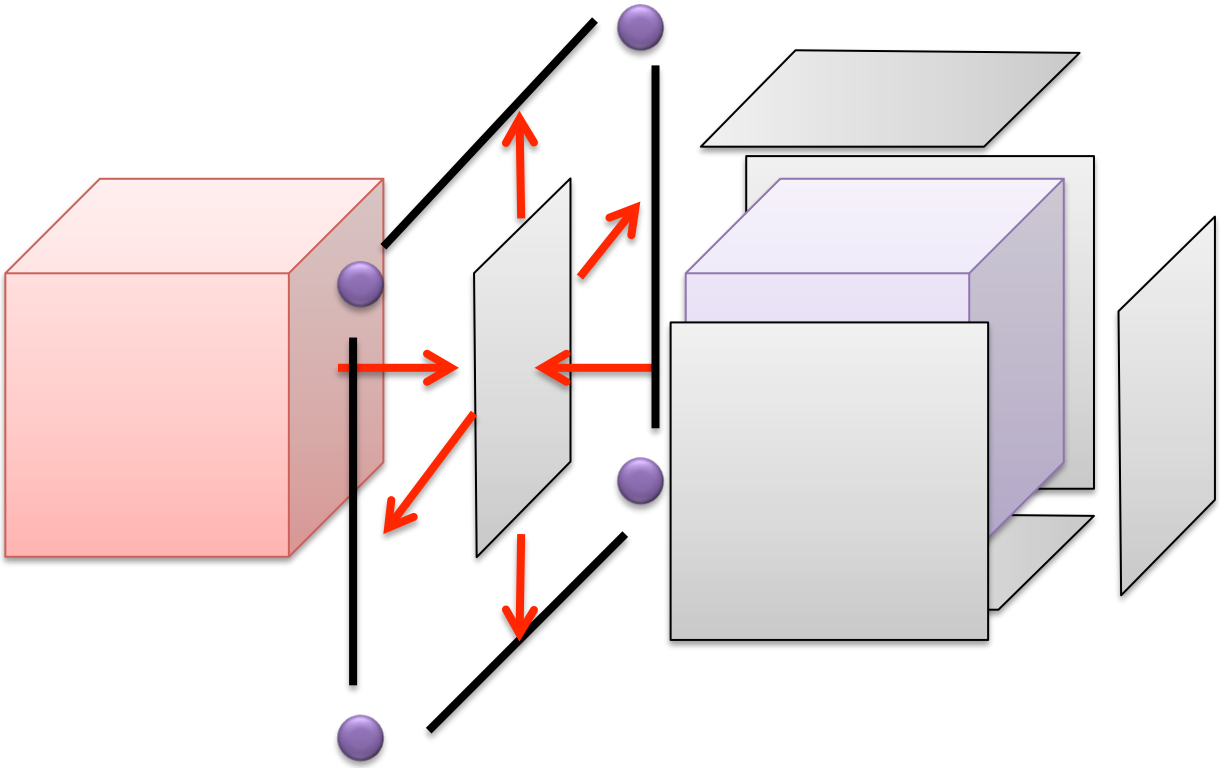



# Un graphe en soi : exemple bis



# Simplicial Complex



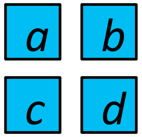
**Brep**
**Gmap**
**...**

# Relational spaces

- A space is a closed world
- Each point in space is an observer of the other points
- Each point has its identity from the relationships it has with the other points
- This is not far from the concept of monad in Leibniz

# Towards a formalization

- Which mathematical object may specify the internal structure of the points?



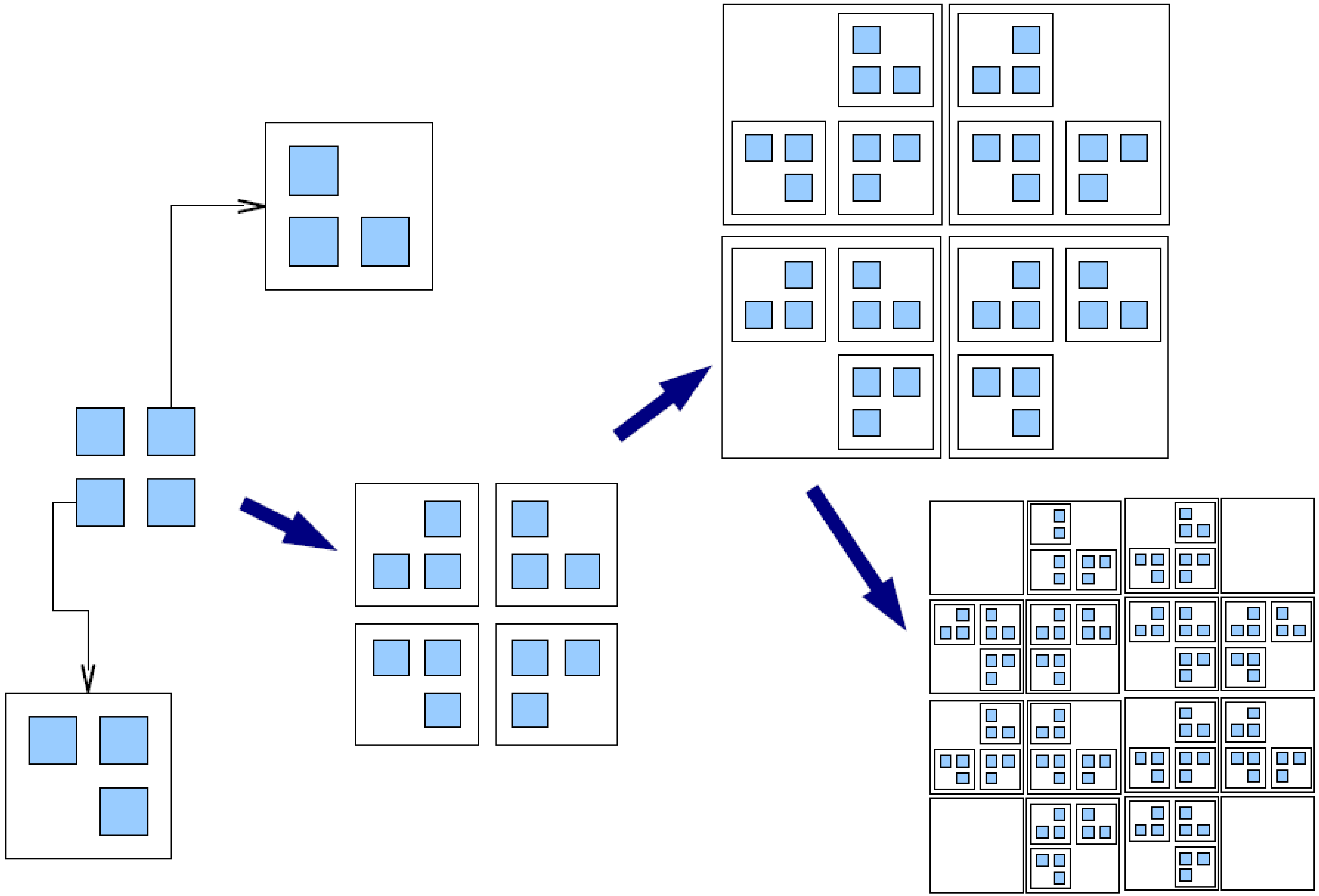

$$\begin{cases} a = \{\, b, d, c \,\} \\ b = \{\, d, c, a \,\} \\ c = \{\, a, b, d \,\} \\ d = \{\, c, a, b \,\} \end{cases}$$

- We need **multisets** because the equations are symmetric for all variable permutations and so $a = b = c = d$

- In fact we need more: a surface (not a graph, even if a surface can be “coded” by graph, cf. V–V system)

- But it is enough for a first approach

# A 4-point space

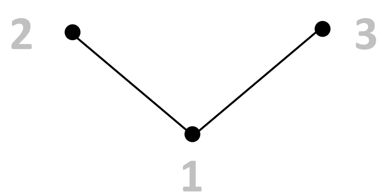


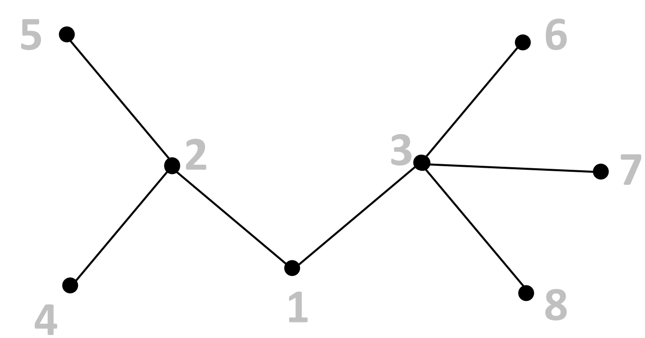
# Graphe de variété maximale

- Dans un « GBF » (graphe de Cayley) tous les points sont indistingables (il y a un **automorphisme** qui transforme un point en n'importe quel autre)

- Suivant Leibniz : tous les états indistingables sont identifiés

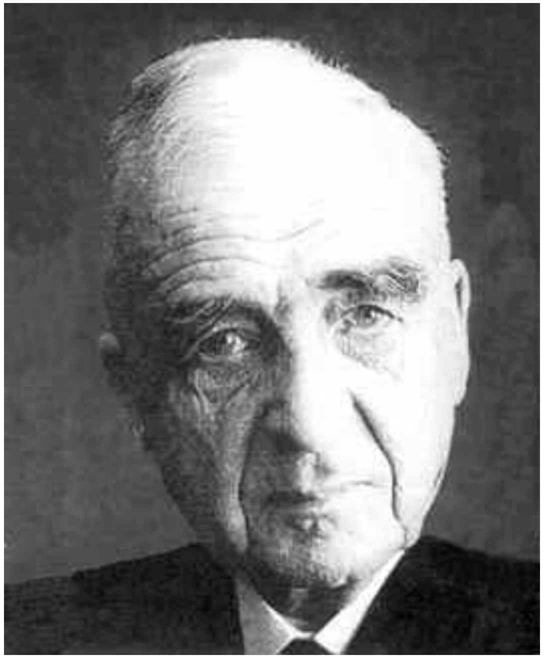
- Barbour et Smolin se sont intéressés aux graphes de variété maximale dans le contexte de la physique (un tel graphe = un état intrinsèque de l'univers).
  On peut même définir des plus maximaux que d'autres avec le diamètre (l'horizon) nécessaire à distinguer les sommets.



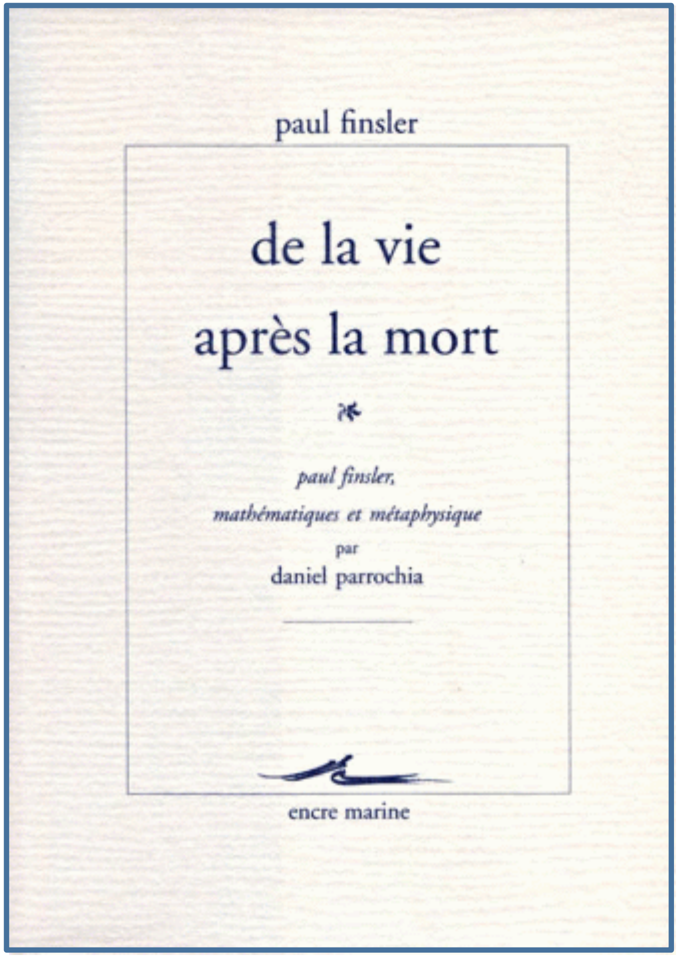



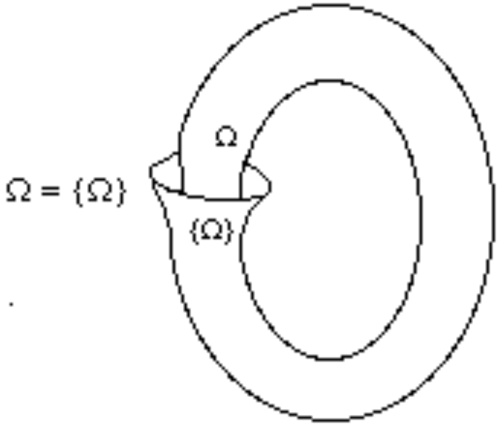
# A metaphysical conclusion

# Les ensembles non-standards via Paul Finsler



Paul Finsler (1894 -1970)




- Ses travaux sur les ensembles circulaires l'invite à *identifier un élément à une classe* et, en l'occurrence, chaque homme à l'humanité tout entière.

- Ses travaux sur les espaces de Riemann lui montrent que *le fini n'est pas nécessairement limité*.

- De sorte qu'il imagine que **l'autre vie n'est que la vie des autres.**