A Webster-Lokshin model for waves with viscothermal losses and impedance boundary conditions: strong solutions.

Houssem Haddar[1], Thomas Hélie[2], and Denis Matignon[1*]

[1] INRIA, domaine de Voluceau-Rocquencourt, B.P. 105,
F-78153 Le Chesnay, France

[2] IRCAM, Analysis-Synthesis team, 1, place Igor Stravinsky,
F-75 004 Paris, France

Abstract. Acoustic waves travelling in a duct with viscothermal losses at the wall and radiating conditions at both ends obey a Webster-Lokshin model that involves fractional time-derivatives in the domain and dynamical boundary conditions. This system can be interpreted as the coupling of three subsystems: a wave equation, a diffusive realization of the pseudo-differential time-operator and a dissipative realization of the impedance, thanks to the Kalman-Yakubovich-Popov lemma.
Existence and uniqueness of strong solutions of the system are proved, using the Hille-Yosida theorem.

1 Introduction

The Lokshin model originally presented in [6] and referred to in [4] in a half-space has then been derived in a bounded space in [11], and modified in [5].
In the case of constant-coefficients, it has been solved analytically and analyzed in both time-domain and frequency-domain in [9], while the principle of an energy analysis has been given in [8].
The problem at stake here is in a bounded domain, with non-constant coefficients (due to Webster equation for horns and space-varying coefficients for the viscothermal effects). Existence and uniqueness of strong solutions of the free evolution problem is proved in an energy space (see also [13]), and the coupling between passive subsystems is used as main method of analysis, as in [7, ch. 5].
We begin with the formulation of the problem in §2; a key point is the reformulation as coupled first-order systems in §3, thanks to diffusive realizations of fractional differential operators; the analysis of the global system follows in §4.
The following slight extensions or wider perspectives are in view:

- existence of weak solution of a variational formulation of the coupled problem; uniqueness with dynamical boundary conditions of any order?
- numerical analysis of the variational problem;
- use some infinite-dimensional analogue of the KYP lemma for some more realistic impedances, as in [3];

* on sabbatical leave from ENST, TSI dept. & CNRS, URA 820. 46, rue Barrault F-75634 Paris Cedex 13, FRANCE.

- study of the boundary-controlled equation;
- proof of precompactness of strong trajectories in the energy space, following [7, ch. 3], in order to apply LaSalle's invariance principle.

2 Mathematical formulation of the model

Both models from [11] or [5] are of the following type:

$$\partial_t^2 \phi + \eta(z)\, \partial_t^{3/2} \phi + \varepsilon(z)\, \partial_t^{1/2} \phi - \frac{1}{r^2} \partial_z (r^2\, \partial_z \phi) = 0 \tag{1}$$

where $z \in [0,\, 1]$ is the space variable, $r, \varepsilon, \eta \in L^\infty(0,1;\mathbb{R}^+)$ and the radius of the duct fulfills $r \geq r_0 > 0$; dynamical boundary conditions are associated with (1). We prefer to work on first order systems in the $(p,\, v)$ variables:

$$\partial_t p = -\frac{1}{r^2}\partial_z(r^2\, v) - \varepsilon\, \partial_t^{-1/2} p - \eta\, \partial_t^{1/2} p\,, \tag{2a}$$

$$\partial_t v = -\partial_z p\,, \tag{2b}$$

$$\hat{p}_i(s) = \mp \mathcal{Z}_i(s)\, \hat{v}_i(s) \quad \text{for } i = 0, 1\,. \tag{2c}$$

The boundary conditions (2c) at $z = i$ are formulated in the Laplace domain, with shorthand notation $p_i(t) = p(z = i,\, t)$; the impedances $\mathcal{Z}_i(s)$ are strictly positive real, i.e. $\Re e(\mathcal{Z}_i(s)) > 0$, $\forall s$, $\Re e(s) \geq 0$.

3 A coupled formulation

3.1 Dissipative realizations for positive-real impedances (Kalman-Yakubovich-Popov lemma)

For a strictly positive real impedance $\mathcal{Z}_i(s)$ of rational type, we choose a *minimal* realization $(A_i,\, B_i,\, C_i,\, d_i)$ with state x_i of *finite* dimension n_i ($A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, $C_i \in \mathbb{R}^{1 \times n_i}$ and $d_i \in \mathbb{R}$); then, following e.g. [1,12], there exists $P_i \in \mathbb{R}^{n_i \times n_i}$, $P_i = P_i^T > 0$, such that the following energy balance holds:

$$\int_0^T p_i(t)\, v_i(t)\, dt = \frac{1}{2}(x_i^T(T)\, P_i\, x_i(T)) + \frac{1}{2}\int_0^T \left(x_i^T(t)\ v_i(t)\right) \mathcal{M}_i \begin{pmatrix} x_i(t) \\ v_i(t) \end{pmatrix} dt\,,$$

with $\mathcal{M}_i = \begin{pmatrix} -A_i^T P_i - P_i A_i & C_i^T - P_i B_i \\ C_i - B_i^T P_i & 2 d_i \end{pmatrix} \geq 0.$

3.2 Dissipative diffusive realizations for positive pseudo-differential time-operators

We refer to [13, § 5.] for the treatment of completely monotone kernels, and [10] and references therein for links between diffusive representations and fractional differential operators.
For a positive measure M on $\mathbb{R}^+$, such that $c_1(M) = \int_0^{+\infty} \frac{dM(\xi)}{1+\xi} < +\infty$, we define: $H_M = L^2(\mathbb{R}^+, dM)$, $V_M = L^2\,(\mathbb{R}^+, (1+\xi)\, dM)$, and $\widetilde{H}_M = L^2(\mathbb{R}^+, \xi\, dM)$. The spaces H_N, V_N and $\widetilde{H}_N$ are defined analogously for a positive measure N.

First diffusive representations. Consider the dynamical system with input $p \in L^2(0,T)$ and output $\theta \in L^2(0,T)$:

$$\partial_t \varphi(\xi,t) = -\xi\, \varphi(\xi,t) + p(t) \quad \text{with} \quad \varphi(\xi,0) = 0 \quad \forall\, \xi \in \mathbb{R}^+ \,, \tag{3a}$$

$$\theta(t) = \int_0^{+\infty} \varphi(\xi,t)\, dM(\xi) \,. \tag{3b}$$

As an example, $dM_\beta(\xi) = \mu_\beta(\xi)\, d\xi$ with density $\mu_\beta(\xi) = \frac{\sin(\beta\pi)}{\pi}\, \xi^{-\beta}$ provides a diffusive realization of the fractional integral I^β, a pseudo-differential time-operator, the symbol of which is $s^{-\beta}$, with $0 < \beta < 1$; it realizes $\theta = I^\beta p = \partial_t^{-\beta} p$. The following energy balance will be useful:

$$\int_0^T p(t)\, \theta(t)\, dt = \frac{1}{2} \int_0^{+\infty} \varphi(\xi,T)^2\, dM + \int_0^T \int_0^{+\infty} \xi\, \varphi(\xi,t)^2\, dM\, dt \,. \tag{4}$$

Extended diffusive representations. Consider the dynamical system with input $p \in H^1(0,T)$ and output $\widetilde{\theta} \in L^2(0,T)$:

$$\partial_t \widetilde{\varphi}(\xi,t) = -\xi\, \widetilde{\varphi}(\xi,t) + p(t) \quad \text{with} \quad \widetilde{\varphi}(\xi,0) = 0 \quad \forall\, \xi \in \mathbb{R}^+ \,, \tag{5a}$$

$$\widetilde{\theta}(t) = \int_0^{+\infty} \partial_t \widetilde{\varphi}(\xi,t)\, dN(\xi) = \int_0^{+\infty} [p(t) - \xi\, \widetilde{\varphi}(\xi,t)]\, dN(\xi) \,. \tag{5b}$$

As an example, $dN_{1-\alpha}(\xi) = \mu_{1-\alpha}(\xi)\, d\xi$ with density $\mu_{1-\alpha}(\xi)$ provides a diffusive realization of the fractional derivative D^α, a pseudo-differential time-operator, the symbol of which is s^α, with $0 < \alpha < 1$; it realizes $\widetilde{\theta} = D^\alpha p = \partial_t^\alpha p$.
The following energy balance will be useful:

$$\int_0^T p(t)\, \widetilde{\theta}(t)\, dt = \frac{1}{2} \int_0^{+\infty} \xi\, \widetilde{\varphi}(\xi,T)^2\, dN + \int_0^T \int_0^{+\infty} (p - \xi\, \widetilde{\varphi})^2\, dN\, dt \,. \tag{6}$$

3.3 An abstract formulation

Thus, the global system (2a)–(2c) can be put in the abstract form $\partial_t X + \mathcal{A}\, X = 0$, where:

$$\mathcal{A} \begin{pmatrix} x_0 \\ x_1 \\ p \\ v \\ \varphi \\ \widetilde{\varphi} \end{pmatrix} = \begin{pmatrix} -A_0 x_0 - B_0\, v(z=0) \\ -A_1 x_1 - B_1\, v(z=1) \\ \frac{1}{r^2} \partial_z (r^2\, v) + \varepsilon \int_0^{+\infty} \varphi\, dM + \eta \int_0^{+\infty} [p - \xi\, \widetilde{\varphi}]\, dN \\ \partial_z p \\ \xi \varphi - p \\ \xi \widetilde{\varphi} - p \end{pmatrix} ; \tag{7}$$

together with the boundary conditions $p(z=0) = -C_0 x_0 - d_0\, v(z=0)$ and $p(z=1) = C_1 x_1 + d_1\, v(z=1)$. In the sequel, we shall analyze the well-posedness of this system.

4 Well-posedness of the global system

4.1 Functional spaces

Let $L^2_{r^2} = L^2(0,1;r^2(z)\,dz)$, $H^1_p = \left\{p \in L^2_{r^2},\ \int_0^1 \left[p^2 + (\partial_z p)^2\right]\ r^2(z)\,dz < +\infty\right\}$ and $H^1_v = \left\{v \in L^2_{r^2}, \int_0^1 r^2(z)\,v^2 + \frac{1}{r^2(z)}[\partial_z(r^2 v)]^2\,dz < +\infty\right\}$.
The natural *energy space* is the following Hilbert space:

$$\mathcal{H} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times L^2_{r^2} \times L^2_{r^2} \times L^2(0,1;H_M;\varepsilon\,r^2\,dz) \times L^2(0,1;\widetilde{H}_N;\eta\,r^2\,dz)\ ,$$

with scalar product for $X = (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T$ and $Y = (y_0, y_1, q, w, \psi, \widetilde{\psi})^T$:

$$\begin{aligned}(X,\,Y)_{\mathcal{H}} &= r_0^2(x_0^T P_0 x_0) + r_1^2(x_1^T P_1 x_1) + (p,\,q)_{L^2_{r^2}} + (v,\,w)_{L^2_{r^2}} \\ &\quad + \int_0^1 (\varphi,\,\psi)_{H_M}\varepsilon(z)\,r^2(z)\,dz + \int_0^1 (\widetilde{\varphi},\,\widetilde{\psi})_{\widetilde{H}_N}\eta(z)\,r^2(z)\,dz\ . \qquad (8)\end{aligned}$$

We define the space $\mathcal{V} \subset \mathcal{H}$:

$$\mathcal{V} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times H^1_p \times H^1_v \times L^2(0,1;V_M;\varepsilon\,r^2\,dz) \times L^2(0,1;\widetilde{H}_N;\eta\,r^2\,dz)\ ,$$

and the domain of operator $\mathcal{A}$ by:

$$D(\mathcal{A}) = \left\{ (x_0, x_1, p, v, \varphi, \widetilde{\varphi})^T \in \mathcal{V},\ \left| \begin{array}{l} p(z=0) = -C_0 x_0 - d_0\,v(z=0) \\ p(z=1) = C_1 x_1 + d_1\,v(z=1) \\ (p - \xi\widetilde{\varphi}) \in L^2(0,1;V_N;\eta\,r^2\,dz) \end{array} \right. \right\}\ .$$

4.2 Existence and uniqueness of strong solutions (Hille-Yosida theorem), regularity results and energy equalities

Theorem 1.

$\forall X_0 \in D(\mathcal{A}), \exists! X(t) \in C^1([0,+\infty[;\mathcal{H}) \cap C^0([0,+\infty[;D(\mathcal{A}))$, *such that* $\partial_t X + \mathcal{A}\,X = 0$ *on* $[0,+\infty[$, *and* $X(0) = X_0$.

This solution satisfies $\frac{d}{dt}\left\{\frac{1}{2}\|X(t)\|^2_{\mathcal{H}}\right\} = -(\mathcal{A}X(t),\,X(t))_{\mathcal{H}} \leq 0$.

Proof. The *monotonicity* of $\mathcal{A}$ follows from the identity: $\forall X \in D(\mathcal{A})$,

$$\begin{aligned}(\mathcal{A}X,\,X)_{\mathcal{H}} &= \frac{r_0^2}{2}\left(x_0^T\ v(0)\right)\mathcal{M}_0\begin{pmatrix}x_0\\ v(0)\end{pmatrix} + \frac{r_1^2}{2}\left(x_1^T\ v(1)\right)\mathcal{M}_1\begin{pmatrix}x_1\\ v(1)\end{pmatrix} \\ &\quad + \int_0^1 \|\varphi\|^2_{\widetilde{H}_M}\varepsilon\,r^2\,dz + \int_0^1 \|p - \xi\,\widetilde{\varphi}\|^2_{H_N}\eta\,r^2\,dz\ . \qquad (9)\end{aligned}$$

Hence $(\mathcal{A}X,\,X)_{\mathcal{H}} \geq 0$, $\forall X \in D(\mathcal{A})$.
The details of the proof of *maximality* of $\mathcal{A}$ will be skipped. For a given $Y \in \mathcal{H}$, seeking for $X \in D(\mathcal{A})$ solution of $(I + \mathcal{A})X = Y$ can be done through the following steps:

1. solve the algebraïc part with respect to x_0, x_1 (requires $s = 1 \notin \mathrm{spec} A_i$);
2. solve the algebraïc part with respect to $\varphi, \widetilde{\varphi}$ and check the functional spaces;
3. solve the differential system (I) with respect to (p, v), which leads to a regular Sturm-Liouville problem with boundary conditions:
   (a) test this equation in (p, v) with $(q, w) \in H^1_p \times L^2_{r^2}$,
   (b) set $w = \partial_z q$ and deduce a variational formulation for $(p, q) \in H^1_p \times H^1_p$,
   (c) apply Lax–Milgram theorem (see [2, ch. VIII]) (requires $\mathcal{Z}_i(s = 1) > 0$),
   (d) show that solution p of the variational problem, together with v defined adequately from p, are solutions of the initial differential system (I), belong to the appropriate spaces and fulfill the boundary conditions.

From the monotonicity and maximality of operator $\mathcal{A}$, we conclude with Hille-Yosida theorem. □

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