REPRESENTATION OF THE WEAKLY NONLINEAR PROPAGATION IN AIR-FILLED PIPES WITH VOLTERRA SERIES

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Abstract— *This article presents the resolution of a boundary controlled nonlinear partial differential equation with Volterra series. The generalized Burgers'equation which is investigated, accounts for the dominant effects of the acoustic propagation in cylindrical ducts involved in brass musical instruments.*

I. INTRODUCTION

The Volterra series give a systematic representation for a wide class of nonlinear systems, including linear differential systems, memory-less nonlinear functions, and their combinations [1]. Practically, they are very attractive for weakly nonlinear ordinary differential equations for which keeping only the low-order kernels yields good approximations.

In this article, the Volterra series are used in the more general context of a boundary controlled nonlinear partial differential equation. Their formal framework gives an interesting alternative to the perturbation method usually used. The only difference from the case of ordinary differential equations concerns the identification of the kernels: each one is performed by solving a linear differential equation and not through an algebraic resolution.

The intended application deals with the acoustic propagation of planar waves in cylindrical ducts which induce thermo-viscous losses on the wall. In such a case, the nonlinearity of the propagation may lead to a shock-wave after a sufficiently long distance. For shorter distances, the weaker distortion is still audible for waves with high amplitudes, and is involved in brass instruments. Indeed, the so-called "brass effect" exactly denotes the brightness of the sound obtained at *fortissimo*, due to this distortion.

The practical interest of using Volterra series is that the obtained input-output system allows the simulation of stationary waves as well as transients : standing waves or periodic inputs are not required as methods such as the harmonic balance do [2].

In section II, the investigated Burgers'acoustic model is presented. The section III installs definitions, notations, and some properties on the Volterra series. Then, the section IV establishes the analytical method used to solve the acoustic partial differential equation finding all the Volterra kernels. Finally, the section V presents analytical results in the time domain for sinusoidal inputs, and perspectives for a general low-cost time-realization of the Burgers'system.

II. ACOUSTIC MODEL

Let the massic density, the speed of the sound, the atmospheric pressure, the specific heat ratio, the kinematic viscosity, and the Prandtl number for the air be $\rho_0 \approx 1.2\,\mathrm{Kg.m}^{-3}$, $c_0 \approx 344\,\mathrm{m.s}^{-1}$, $P_0 \approx 1.013\,10^5$ Pa, $\gamma \approx 1.4$, $\nu \approx 1.5\,10^{-5}$, and $P_r \approx 0.7$, respectively.

In [2], the acoustic waves propagating in a cylindrical pipe are shown to be well described via two adimensionnal progressive planar waves q^+ and q^- by

- massic density:
$$\rho_a = \rho_0 \left[q^+ + q^- + O(M^2)\right]$$
- longitudinal particle velocity:
$$u_a = c_0 \left[q^+ - q^- + O(M^2)\right]$$
- pressure:
$$p_a = P_0\,\gamma \left[q^+ + q^- + O(M^2)\right]$$

where M denotes the order of magnitude of q^+, q^-, and of the Mach number u_a/c_0.

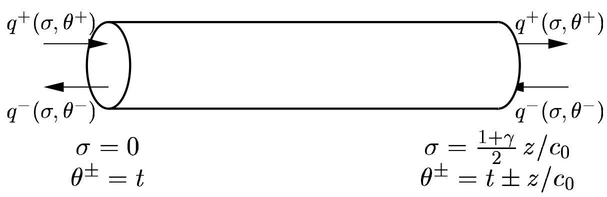

Fig. 1. Progressive waves propagating in a cylindrical pipe

These waves satisfy the generalized Burgers'equations

$$\xi\,\partial_\sigma q \;=\; q\,\partial_\theta q \;-\; \alpha_0\,\partial_\theta^{\frac{1}{2}} q, \tag{1}$$

with $\xi = +1$ and $\theta = t - z/c_0$ for $q = q^+$, $\xi = -1$ and $\theta = t + z/c_0$ for $q = q^-$, and $\sigma = \frac{1+\gamma}{2}\,z/c_0$. The coefficient $\alpha_0 = \frac{2}{R_0}\kappa_0$ with $\kappa_0 = \frac{\sqrt{\nu}(\sqrt{P_r}+\gamma-1)}{\sqrt{P_r}(\gamma+1)} \approx 2.39\,10^{-3}$ accounts the visco-thermal losses for a pipe with the radius R_0. The operator $\partial_\theta^{\frac{1}{2}}$ is the fractional derivative of order $1/2$ for causal functions associated to the Laplace symbol $s^{\frac{1}{2}}$ on $\boldsymbol{C}$ with a cut on $\boldsymbol{R}^-$.

The system defined by FIG. 1 and Eq. (1) is weakly nonlinear if $q\,\partial_\theta q$ is not greatly solicited. This occurs for small amplitudes and low frequencies, but also if this nonlinearity is integrated over a short length, *i.e.* for small σ. For brass instruments, these features make the nonlinearity weak but usually not negligible, so that Volterra series may have a practical interest.

III. VOLTERRA SERIES : DEFINITIONS AND PROPERTIES

A. Definitions and notations

A system **(S)** is described by a Volterra series of kernels $\{h_n\}_{n\in\boldsymbol{N}^*}$ for inputs $|u(t)| < \rho$ if and only if the output $y(t)$ is given by the multi-convolutions

$$y(t)=\sum_{n=1}^{+\infty}\int\cdots\int_{-\infty}^{+\infty} h_n(\tau_1,\ldots,\tau_n)\,u(t\text{-}\tau_1)\ldots u(t\text{-}\tau_n)\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n, \quad (2)$$

where ρ is the convergence radius of $\sum_{n=1}^{+\infty}\|h_n\|_1 x^n$, with $\|h_n\|_1 = \int\cdots\int_{-\infty}^{+\infty}|h_n(\tau_1,\ldots,\tau_n)|\,\mathrm{d}\tau_1\ldots\mathrm{d}\tau_n$. For causal systems, h_n is zero for $\tau_k < 0$.

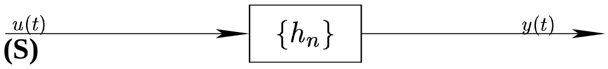

Fig. 2. System **(S)** represented by Volterra kernels.

For causal systems, the mono-lateral [3, (29.1.2)] Laplace transform of $h_n(\tau_1,\ldots,\tau_n)$ is noted $H_n(s_1,\ldots,s_n)$. It is is analytic for $s_k, \Re\mathrm{e}(s_k) > 0$.

B. Interconnection laws

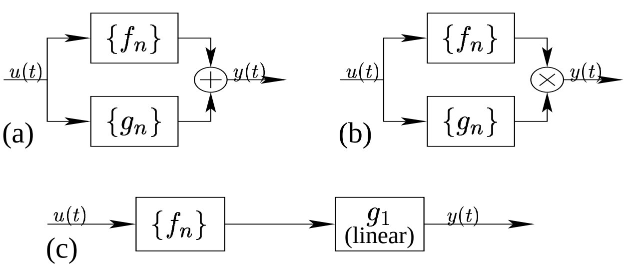

Fig. 3. Sum (a), product (b), and cascade (c) of two systems

The kernels $\{H_n\}$ of the systems (a), (b), and (c) are given respectively by [4, p. 34,35]

$$H_n(s_1,\ldots,s_n) = F_n(s_1,\ldots,s_n) + G_n(s_1,\ldots,s_n), \quad (3)$$

$$H_n(s_1,\ldots,s_n) = \sum_{p=1}^{n-1} F_p(s_1\ldots,s_p)G_{n-p}(s_{p+1},\ldots,s_n), \quad (4)$$

$$H_n(s_1,\ldots,s_n) = F_n(s_1,\ldots,s_n)\,G_1(s_1+\ldots+s_n). \quad (5)$$

IV. MODELING OF THE PROPAGATION WITH VOLTERRA KERNELS

The problems for q^+ and q^- are symmetrical for $z \mapsto -z$ so that it is assumed that $q = q^+$ and $\xi = +1$, in the following,

Note that recursive calculations by the perturbation method could be adapted without explicitly invoking Volterra series and yield similar results [5, p.207-209]. However, the Volterra series framework gives a systematic and straightforward approach which justifies its extension to the case of partial differential equations.

A. Deriving the equations satisfied by the kernels

Let the system **(S)** give the output $y^{(\sigma)}(\theta) = q(\sigma,\theta)$ from the input $u(\theta) = q(0,\theta)$ where q is governed by Eq. (1). Let $\{h_n^{(\sigma)}(\tau_1,\ldots,\tau_n)\}_{n\in\boldsymbol{N}^*}$ be the σ-parameterized kernels of **(S)**. By definition, this system is the identity for $\sigma = 0$ so that

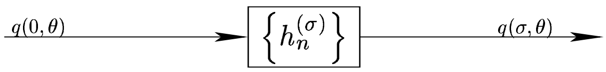

Fig. 4. Definition of the Burgers'kernels

$$H_1^{(0)}(s_1) = 1, \quad (6)$$

$$H_n^{(0)}(s_1,\ldots,s_n) = 0, \qquad \forall n \geq 2. \quad (7)$$

From Eq. (1), it appears that the systems **(S1)** and **(S2)** are equivalent (see FIG. 5).

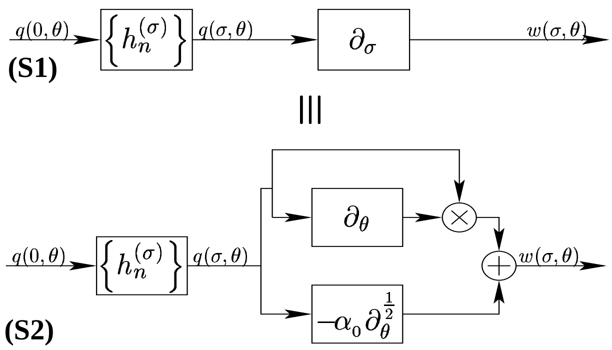

Fig. 5. Equivalent systems for a wave q governed by Eq. (1)

As the linear operator ∂_σ does not depend on θ, the kernels of **(S1)** in the Laplace domain are $\partial_\sigma H_n^{(\sigma)}$.

The operators ∂_θ and $-\alpha_0\partial_\theta^{\frac{1}{2}}$ are associated to the linear kernels $F_1(s) = s$ and $G_1(s) = -\alpha_0 s^{\frac{1}{2}}$ respectively. The interconnection laws Eq. (3-5) yield straightforwardly the kernels of **(S2)**.

Finally, the equality of the kernels of **(S1)** and **(S2)** is obtained by

$$\begin{aligned}\partial_\sigma H_n^{(\sigma)}(s_1,\dots,s_n) \;=\; & \\ \sum_{p=1}^{n-1}(s_1+\dots+s_p)\, H_p^{(\sigma)}(s_1,\dots,s_p)\, H_{n-p}^{(\sigma)}(s_{p+1},\dots,s_n) & \\ -\,\alpha_0\sqrt{s_1+\dots+s_n}\; H_n^{(\sigma)}(s_1,\dots,s_n), & \qquad (8)\end{aligned}$$

which gives a linear ordinary differential equation to solve for each kernel $H_n^{(\sigma)}$.

B. Explicit solutions of the Volterra Kernels

Equations (6-8) make the resolution of the kernels possible, as described below.

B.1 Linear kernel

For $n = 1$, Eq. (8) is

$$\partial_\sigma H_1^{(\sigma)}(s_1) = -\,\alpha_0\sqrt{s_1}\; H_1^{(\sigma)}(s_1), \qquad (9)$$

so that the solution which satisfies Eq. (6) is

$$H_1^{(\sigma)}(s_1) = \mathrm{e}^{-\alpha_0\,\sigma\sqrt{s_1}},\;\; \Re\mathrm{e}(s_1) > 0. \qquad (10)$$

B.2 Second order kernel

For $n = 2$, Eq. (8) becomes

$$\begin{aligned}\partial_\sigma H_2^{(\sigma)}(s_1,s_2) \;=\; & -\,\alpha_0\sqrt{s_1+s_2}\; H_2^{(\sigma)}(s_1,s_2) \\ & + s_1\mathrm{e}^{-\alpha_0\,(\sqrt{s_1}+\sqrt{s_2})\,\sigma}. \qquad (11)\end{aligned}$$

The solution which satisfies Eq. (7) is obtained

$$H_2^{(\sigma)}(s_1,s_2) = \frac{s_1}{\alpha_0}\,\frac{\mathrm{e}^{-\alpha_0\,\sigma\sqrt{s_1+s_2}} - \mathrm{e}^{-\alpha_0\,\sigma\,(\sqrt{s_1}+\sqrt{s_2})}}{\sqrt{s_1}+\sqrt{s_2}-\sqrt{s_1+s_2}}. \qquad (12)$$

B.3 Higher order kernels

By recurrence, it may be proven that the kernels $H_n^{(\sigma)}$ take the form ($\Re\mathrm{e}(s_p) > 0$, $\forall p$),

$$H_n^{(\sigma)}(s_1,\dots,s_n) = \sum_{\phi\in\mathbf{S}^n} Q_\phi(s_1,\dots,s_n)\,\mathrm{e}^{-\alpha_0\,\sigma\phi[s_1,\dots,s_n]}, \qquad (13)$$

where $(Q_\phi(s_1,\dots,s_n))_{\phi\in\cup_{n\in\boldsymbol{N}}\mathbf{S}^n}$ are rational functions of square-roots of sums of $(s_k)_{1\le k\le n}$, $\mathbf{S}^1 = \{\sqrt{\cdot}\}$, and $\mathbf{S}^n = \{\oplus,+\}^{n-1}$ ($\forall n \ge 2$) with the convention $\phi = (\oplus,+,+,\dots,+,\oplus) \in \mathbf{S}^n \Rightarrow \phi[s_1,\dots,s_n] = \sqrt{s_1+s_2}+\sqrt{s_3}+\dots+\sqrt{s_{n-1}+s_n}$. In other words, $\oplus$ is an addition under the square root and $+$ outside the square-root.

From Eq. (7) and Eq. (13), it follows that

$$Q_{\phi_{n,\oplus}}(s_1,\dots,s_n) = -\sum_{\phi\in\mathbf{S}^n\setminus\{\oplus\}^{n-1}} Q_\phi(s_1,\dots,s_n), \qquad (14)$$

where $\phi_{n,\oplus}$ denotes the element $(\oplus,\oplus,\dots,\oplus) \in \{\oplus\}^{n-1} \subset \mathbf{S}^n$.

Moreover, substituting $H_n^{(\sigma)}$ in Eq. (8) by their expressions Eq. (13), and identifying the terms for each exponential $\mathrm{e}^{-\alpha_0\sigma\phi[s_1,\dots,s_n]}$ with $\phi \in \mathbf{S}^n \setminus \{\oplus\}^{n-1}$, the recurrent equation on the rational functions is obtained. This yields, $\forall\phi \in \mathbf{S}^n \setminus \{\oplus\}^{n-1}$,

$$Q_\phi(s_1,\dots,s_n) = \frac{\displaystyle\sum_{p\in\mathcal{I}^+(\phi)}(s_1+\dots+s_p)\, Q_{(\phi_k)_{1\le k\le p-1}}(s_1,\dots,s_p)\, Q_{(\phi_k)_{p+1\le k\le n-1}}(s_{p+1},\dots,s_n)}{\alpha_0\left(\sqrt{s_1+\dots+s_n}-\phi[s_1,\dots,s_n]\right)}, \qquad (15)$$

where $\mathcal{I}^+(\phi) = \{p\;/\;\phi_p = +\}$ and, following the convention adopted for $\mathbf{S}^1$, $Q_{(\phi_k)_{k\in\emptyset}}(s_1) \equiv Q_{\sqrt{\cdot}}(s_1) = 1$ (see Eq. (10)).

C. Remarks on the effects of visco-thermal losses

The visco-thermal losses are expressed through α_0. When this effect is missing ($\alpha_0 \to 0$), the first kernels becomes $H_1^{(\sigma)}(s_1) \sim 1$ and $H_2^{(\sigma)}(s_1,s_2) \sim \sigma\, s_1$, leading to well-known properties :

$H_1^{(\sigma)}$*:* a traveling planar wave is not scattered by the linear propagation in a cylindrical pipe,

$H_2^{(\sigma)}$*:* the wave distortion increases with σ, leading to the formation of the shock-wave with a “waterfall”.

The visco-thermal losses modify the asymptotic behavior and yield a correction on this last “physical singularity”. Indeed, $\lim_{\sigma\to+\infty}|H_2^{(\sigma)}(s_1,s_2)| = 0$, $\forall s_1, s_2$, $\Re\mathrm{e}(s_1) > 0, \Re\mathrm{e}(s_2) > 0$, and more generally, $\forall n$, $\forall\phi \in \mathbf{S}^n$,

$$\Re\mathrm{e}(s_1,\dots,s_n) > 0 \Rightarrow \lim_{\sigma\to+\infty}\mathrm{e}^{-\alpha_0\sigma\phi[s_1,\dots,s_n]} = 0,$$

so that the distortion decreases after a sufficiently long distance.

V. Time-simulation

A. Case of periodic inputs

For a periodic signal $u(\theta) = \sum_{k=-\infty}^{+\infty} c_k \mathrm{e}^{\mathrm{i}k\omega\theta}$, the response of the system takes the particular form $y(t) = \sum_{k=-\infty}^{+\infty} d_k \mathrm{e}^{\mathrm{i}k\omega\theta}$ where [4, (3.104)]

$$d_k = \sum_{n=1}^{+\infty}\;\sum_{\substack{k_1,\dots,k_n=-\infty\\ k_1+\dots+k_n=k}}^{+\infty} c_{k_1}\cdots c_{k_n}\, H_n(ik_1\omega,\dots,ik_n\omega). \qquad (16)$$

This yields analytical results (which also can be reached for multi-periodic input signals [4, (3.119)]). This furnishes an alternative method to the harmonic balance used in [2].

To avoid intricate expressions, the calculations are made here for the approximated system $\{H_1^{(\sigma)}, H_2^{(\sigma)}\}$ and for the simple case where $u(\theta) = a\cos(\omega\theta)$ ($c_1 = c_{-1} = a/2$ and $c_k = 0$ else). This yields an output which is the sum of a constant, a fundamental (ω), and a second harmonic (2ω) components. The constant term is

$$d_0 = c_{-1}c_1\left[H_2^{(\sigma)}(i\omega, -i\omega) + H_2^{(\sigma)}(-i\omega, i\omega)\right] = 0. \quad (17)$$

The fundamental component is

$$\begin{aligned} d_{-1}\mathrm{e}^{-\mathrm{i}\omega\theta} + d_1\mathrm{e}^{\mathrm{i}\omega\theta} &= c_{-1}H_1^{(\sigma)}(-i\omega)\mathrm{e}^{-\mathrm{i}\omega\theta} + c_1 H_1^{(\sigma)}(i\omega)\mathrm{e}^{\mathrm{i}\omega\theta} \\ &= a\,\mathrm{e}^{-\alpha_0\sigma\sqrt{\frac{\omega}{2}}}\cos\left(\omega\theta - \alpha_0\sigma\sqrt{\frac{\omega}{2}}\right). \quad (18) \end{aligned}$$

The second harmonic component is

$$\begin{aligned} d_{-2}\mathrm{e}^{-2\mathrm{i}\omega\theta} + d_2\mathrm{e}^{2\mathrm{i}\omega\theta} &= c_{-1}^2 H_2^{(\sigma)}(-i\omega, -i\omega)\mathrm{e}^{-2\mathrm{i}\omega\theta} + c_1^2 H_2^{(\sigma)}(i\omega, i\omega)\mathrm{e}^{2\mathrm{i}\omega\theta} \\ &= \frac{a^2\sqrt{\omega}}{2\alpha_0(2-\sqrt{2})}\left[\mathrm{e}^{-\alpha_0\sigma\sqrt{\omega}}\cos\left(2\omega\theta - \alpha_0\sigma\sqrt{\omega} + \frac{\pi}{4}\right)\right. \\ &\qquad \left. -\mathrm{e}^{-\alpha_0\sigma\sqrt{2\omega}}\cos\left(2\omega\theta - \alpha_0\sigma\sqrt{2\omega} + \frac{\pi}{4}\right)\right]. \quad (19) \end{aligned}$$

Simulations are represented in FIG. 6.

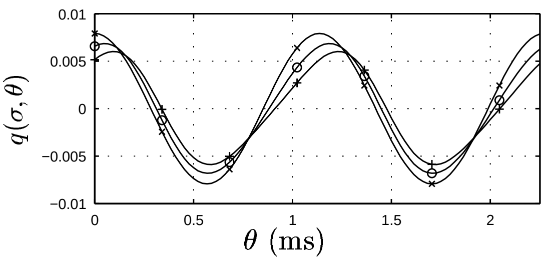

Fig. 6. Waveforms for $a = 7.93\,10^{-3}$ (155 dB SPL), $\omega = 2\pi\,880\,\mathrm{rd.s}^{-1}$, $\alpha_0 = 0.852$ ($R_0 = 5.6$mm), for $\sigma = \frac{1+\gamma}{2}\,\frac{z}{c_0}$ with $z = 0$ (×), $z = 1$m (o), and $z = 2$m (+).

B. Non-stationary case: discussion and perspectives for a low cost time-realization

The main interest of the Volterra approach for the time-simulation is that the output may also be processed for inputs which are neither periodic nor stationary. Practically, the time-realization of each kernel requires that they be approximated by finite order systems, which cannot be straightforwardly obtained here.

Indeed, for the linear kernel, the wave deformation induced by the visco-thermal losses ($\alpha_0 \neq 0$) is well-known. It involves a long memory phenomenon : the impulse response given by [3, (29.3.82)]

$$h_1^{(\sigma)}(\theta) = \frac{\alpha_0\,\sigma}{2\sqrt{\pi\theta^3}}\,\mathrm{e}^{-\frac{\alpha_0{}^2\sigma^2}{4\theta}}\,\mathbf{1}_{\theta>\mathbf{0}}(\theta), \quad (20)$$

is decreasing slower than a damped exponential. This makes the time-simulation of the system heavy or coarse even making use of standard approximation techniques.

Nevertheless, coping with this difficulty is not hopeless: $H_1^{(\sigma)}$ may also be identified to a pseudo-differential operator which admits an extended diffusive representation [6, § 5.2]. This formalism which is not introduced here, makes the derivation of exact infinite time-realizations possible from which accurate low-cost approximations may be deduced [7].

The higher order kernels also clearly involve long memory phenomena induced by the square-root of the Laplace variables. Extending the diffusive representations to multi-variable kernels appears naturally as an attractive new approach to investigate.

VI. CONCLUSION

The Volterra series appears as a relevant tool to solve formally weakly nonlinear partial differential equations. The kernels make the analytical calculation of periodic and stationary solutions possible, furnishing an alternative to the harmonic balance method. But the main interest of this approach is that it is not reduced to such particular solutions : non-stationary inputs may be considered as well.

In the case of the investigated Burgers'equation, the kernels involve long memory phenomena. Previous works on linear systems which hold this property make the diffusive representations and their approximations appear appropriate for building low-cost time-realizations. Their extension to the case of multi-variable kernels could be, with the Volterra series, a new key tool to derive low-cost time-realizations of long-memory weakly nonlinear systems.

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