Computation of convergence radius and error bounds of Volterra series for single input systems with a polynomial nonlinearity

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***Abstract*— In this paper, the Volterra series decomposition of a class of time invariant system, polynomial in the state and affine in the input, with an exponentially stable linear part is analyzed. A formal recursive expression of Volterra kernels of the input-to-state system is derived and the singular inversion theorem is used to prove the non-local-in-time convergence of the Volterra series to a trajectory of the system, to provide an easily computable value for the radius of convergence and to compute a guaranteed error bound for the truncated series. These results are available for infinite norms (Bounded Input Bounded Output results) and also for specific weighted norms adapted to some so-called "fading memory systems" (exponentially decreasing input-output results). The method is illustrated on two examples including a Duffing's Oscillator.**

I. INTRODUCTION

Volterra series is a functional series expansion of the solution of nonlinear controlled systems, first introduced by the Italian mathematician Volterra [1]. This tool has been extensively used in signal processing and control, electronics and electro-magnetic waves, mechanics and acoustics, bio-medical engineering, for modeling, identification and simulation purposes. There exists a vast literature concerning Volterra series. Among others, they were studied in [2], [3], [4], [5] from the geometric control point of view, and in [6], [7], [8] from the input-output representation and realization point of view.

However, only a few results on the convergence are available. In [9], [5], theoretical and local-in-time results are given for control systems, affine in the input, with analytic dynamics and piecewise continuous inputs. Existence results of a convergence radius for continuous inputs are also given in [10] for fading memory systems. More recently, results in the frequency domain have been developed in [11], [12], results relying on regular perturbations (which can be related to Volterra series expansion) are given in [13] and results for quadratic systems have been established in [14].

This paper focuses on the representation of a class of single input (SI) finite dimensional ODE systems into input-to-state Volterra series, and the nonlocal convergence of the series. The systems under consideration are time invariant, polynomial in the state and affine in the input, with an exponentially stable linear part. The convergence analysis is performed for bounded (or possibly, exponentially decreasing) inputs, and for Volterra series converging towards solutions of the system. The main contributions of this paper are to obtain both an easily computable bound of the convergence radius and a guaranteed truncation error bound. The proofs of the results mainly rely on two key ideas: first, a formal recursive expression of Volterra kernels of the input-to-state system is derived; second, the tree-like complexity of the recursive formula makes the use of the singular inversion theorem possible in order to bound the kernels norms and the radius of convergence of the series.

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II. GENERAL FRAMEWORK

A. *Notations and functional setting*

Let $K \in \mathbb{N}^*$ and $(j_1, \ldots, j_K) \in \big(\mathbb{N}^*\big)^K$. Let $\mathbb{E}$, $\mathbb{E}_k$ with $1 \le k \le K$ and $\mathbb{F}$ denote finite-dimensional vector spaces over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|_{\mathbb{E}}$, $\|\cdot\|_{\mathbb{E}_k}$ ($1 \le k \le K$) and $\|\cdot\|_{\mathbb{F}}$, respectively. The following definitions are introduced:

- $\mathcal{L}(\mathbb{E},\mathbb{F})$ is the vector space of continuous linear functions from $\mathbb{E}$ to $\mathbb{F}$ with norm $\|f\|_{\mathcal{L}(\mathbb{E},\mathbb{F})} = \sup_{x\in B_{\mathbb{E}}} \|f(x)\|_{\mathbb{F}}$ where $B_{\mathbb{E}}$ is the unit ball in $\mathbb{E}$.
- $\mathcal{ML}(\mathbb{E}_1, ..., \mathbb{E}_K, \mathbb{F})$ is the vector space of continuous multilinear functions $f : \mathbb{E}_1 \times \ldots \times \mathbb{E}_K \to \mathbb{F}$ with norm $\|f\|_{\mathcal{ML}(\mathbb{E}_1,\ldots,\mathbb{E}_K,\mathbb{F})} = \sup_{(x_1,\ldots,x_K)\in B_{\mathbb{E}_1}\times\ldots\times B_{\mathbb{E}_K}} \|f(x_1, ..., x_K)\|_{\mathbb{F}}$ and $\mathcal{ML}(\underbrace{\mathbb{E}_1,\ldots,\mathbb{E}_1}_{j_1}, \underbrace{\mathbb{E}_2,\ldots,\mathbb{E}_2}_{j_2}, \ldots, \underbrace{\mathbb{E}_K,\ldots,\mathbb{E}_K}_{j_K}, \mathbb{F})$ is denoted $\mathcal{ML}_{j_1,\ldots,j_K}(\mathbb{E}_1, \ldots, \mathbb{E}_K, \mathbb{F})$.

The following function spaces are used in the sequel:

- $\mathcal{L}^1\big(\mathbb{R}_+, \mathbb{E}\big)$ and $\mathcal{L}^\infty\big(\mathbb{R}_+, \mathbb{E}\big)$ are the standard Lebesgue spaces.
- $\mathcal{L}^\infty_{loc}\big(\mathbb{R}_+, \mathbb{E}\big)$ is the set of functions f such that for all compact $\mathcal{K} \subset \mathbb{R}_+$, the restriction of f on $\mathcal{K}$ is in $\mathcal{L}^\infty\big(\mathcal{K}, \mathbb{E}\big)$.
- $\mathcal{B}_{\mathbb{E}}(\lambda)$ for $\lambda \in \mathbb{K}$ is the set of functions f such that $t \mapsto \mathrm{e}^{\lambda t} f(t) \in \mathcal{L}^\infty\big(\mathbb{R}_+, \mathbb{E}\big)$. The norm on $\mathcal{B}_{\mathbb{E}}(\lambda)$ is $\|f\|_{\mathcal{B}_{\mathbb{E}}(\lambda)} = \sup_{t\in\mathbb{R}_+} \Big(\|f(t)\|_{\mathbb{E}}\, \mathrm{e}^{\lambda t}\Big)$. Note that $\mathcal{B}_{\mathbb{E}}(0) = \mathcal{L}^\infty\big(\mathbb{R}_+, \mathbb{E}\big)$, that for all $\lambda \in \mathbb{K}$, $\mathcal{B}_{\mathbb{E}}(\lambda) \subset \mathcal{L}^\infty_{loc}\big(\mathbb{R}_+, \mathbb{E}\big)$ and that, if $\Re\mathrm{e}(\lambda^+) > \Re\mathrm{e}(\lambda^-)$ then $\mathcal{B}_{\mathbb{E}}(\lambda^+) \subset \mathcal{B}_{\mathbb{E}}(\lambda^-)$.
- $\mathcal{V}^N_{\mathbb{E}}(\lambda)$ for $N \in \mathbb{N}^*$ and $\lambda \in \mathbb{K}$ is the set of functions $f : \mathbb{R}_+ \times \mathbb{R}^N_+ \to \mathbb{E}$ such that $t \mapsto \big(\tau \mapsto \mathrm{e}^{\lambda(t-\overline{\tau})} f(t,\tau)\big) \in \mathcal{L}^\infty\big(\mathbb{R}_+, \mathcal{L}^1\big(\mathbb{R}^N_+, \mathbb{E}\big)\big)$, where $\forall \tau = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N_+$, $\overline{\tau} = \tau_1 + \tau_2 + \ldots + \tau_N$. The norm is $\|f\|_{\mathcal{V}^N_{\mathbb{E}}(\lambda)} = \sup_{t\in\mathbb{R}_+} \Big(\mathrm{e}^{\lambda t} \int_{\mathbb{R}^N_+} \|f(t,\tau)\|_{\mathbb{E}}\, \mathrm{e}^{-\lambda\overline{\tau}} \mathrm{d}\tau\Big)$ where $\mathrm{d}\tau$ denotes the Lebesgues measure $\mathrm{d}\tau = \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_N$.

B. System under consideration

The systems under consideration are polynomial systems with zero initial conditions and an affine dependence on the input, that is, for $t \in \mathbb{R}_+$,

$$\begin{aligned} \dot{x} &= f(x,u) = A\,x + B\,u + P(x) + Q(x,u), && (1)\\ y &= g(x,u), && (2)\\ x(0) &= 0, && (3) \end{aligned}$$

where $u : \mathbb{R}_+ \to \mathbb{U} = \mathbb{R}$, $x : \mathbb{R}_+ \to \mathbb{X} = \mathbb{R}^{d_{\mathbb{X}}}$, $y : \mathbb{R}_+ \to \mathbb{R}^{d_{\mathbb{Y}}}$ with finite dimensions $d_{\mathbb{X}} \in \mathbb{N}^*$, $d_{\mathbb{Y}} \in \mathbb{N}^*$, where $A \in \mathcal{M}_{d_{\mathbb{X}}}(\mathbb{R})$ is such that $-a = \max\Big(\Re\mathrm{e}\big(\mathrm{Spec}\,A\big)\Big) < 0$ (stable linear part), $B \in \mathcal{M}_{d_{\mathbb{X}},1}(\mathbb{R})$ is non zero, and where, denoting $J = \deg P$ and $K = \deg Q$, P and Q are expressed as a sum of homogeneous contributions

$$P(x) = \sum_{j=2}^{J} P_j(\underbrace{x,\ldots,x}_{j}), \qquad (4)$$

$$Q(x,u) = \sum_{k=2}^{K} Q_k(\underbrace{x,\ldots,x}_{k-1},u), \qquad (5)$$

with $P_j \in \mathcal{ML}_j(\mathbb{X},\mathbb{X})$ and $Q_k \in \mathcal{ML}_{k-1,1}\big(\mathbb{X},\mathbb{U},\mathbb{X}\big)$. Let us comment on the assumptions on the class of systems considered here

- the stability condition on the linear part of the system is essential to achieve non-local-in-time convergence results,
- the polynomial-in-the-state and affine-in-the-input character of the nonlinear part can be relaxed; all the results given in the paper readily extend to the case of systems, with analytic nonlinearities on the state (see remark 3 below); nevertheless, relaxing the affine assumption on the input is not handled in this paper,
- in the same way, non-zero initial conditions can be dealt with, but we restrict to the simpler case of zero initial conditions here.

C. Volterra series: general definitions and basic properties

The standard definition of Volterra series in [9, p.113] is adapted to the analysis of systems over finite periods $t \in [0,T]$. To tackle convergence results over all the system life ($\mathbb{R}_+$), we introduce the following definition of λ-decreasing Volterra series which makes the analysis over $\mathbb{R}_+$ possible, for systems whose response decrease at least like $\mathrm{e}^{-\lambda t}$ ($\lambda \geq 0$). Then, straightforward extensions of results which are well-known [15] in the BIBO case ($\lambda = 0$) are given.

In the sequel, we consider that $\lambda \in \mathbb{R}_+$, we denote $\mathcal{U} = \mathcal{B}_{\mathbb{U}}(\lambda)$, $\mathcal{X} = \mathcal{B}_{\mathbb{X}}(\lambda)$ and $\mathcal{VS}$ denotes the set of the series $(f_m)_{m\in\mathbb{N}^*}$ such that for all $m \in \mathbb{N}^*$, $f_m \in \mathcal{V}^m$ where $\mathcal{V}^m = \mathcal{V}^m_{\mathbb{X}}(\lambda)$.

Definition 1 (SI λ-decreasing Volterra series): A causal SI-system can be described by a λ-decreasing input-to-state Volterra series if there exist $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}$ and $\rho \in \mathbb{R}_+^*$ such that for all input $u \in \mathcal{U}$ satisfying $\|u\|_{\mathcal{U}} < \rho$, the series

$$x(t) = \sum_{m\in\mathbb{N}^*} \int_{[0,t]^m} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau, \qquad (6)$$

defined for $t \in \mathbb{R}_+$, with $\big[\Pi_m u\big](\tau) = \prod_{i=1}^m u(\tau_i)$, is normally convergent for the norm $\|.\|_{\mathcal{X}}$. For $m \in \mathbb{N}^*$, the function h_m is called the kernel of order m.

Remark 1: The input-to-output Volterra series of the system (1-5) can be deduced from its input-to-state Volterra series by substituting (6) in (2).

Definition 2 (Gain bound function): Let $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}$ be a λ-decreasing Volterra series. Let $\rho \in \mathbb{R}_+$ be the radius of convergence of the formal series $\sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}^m} X^m$. If $\rho > 0$, then the *gain bound function* φ of $\{h_m\}_{m\in\mathbb{N}^*}$ is defined by, $\forall z \in \mathbb{C}$ such that $|z| < \rho$,

$$\varphi(z) = \sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}^m}\, z^m.$$

Theorem 1 (Bounded-input bounded-state relation): Let $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}$ be such that the gain bound function φ has a non zero radius of convergence $\rho > 0$. Then, the Volterra series is convergent in $\mathcal{X}$ for inputs such that $\|u\|_{\mathcal{U}} < \rho$. In this case, $x \in \mathcal{B}_{\mathcal{X}}$ satisfies $\|x\|_{\mathcal{X}} \leq \varphi\big(\|u\|_{\mathcal{U}}\big) < \infty$.

Proof: Let $u \in \mathcal{U}$ be such that $\|u\|_{\mathcal{U}} < \rho$. Then, $\varphi\big(\|u\|_{\mathcal{U}}\big) < \infty$. Now, for all $m \in \mathbb{N}^*$, $\sup_{t\in\mathbb{R}_+}\Big(\big\| \int_{[0,t]^m} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau\big\|_{\mathbb{X}}\,\mathrm{e}^{\lambda t}\Big) \leq \sup_{t\in\mathbb{R}_+}\Big(\int_{[0,t]^m} \big\|\mathrm{e}^{\lambda t} h_m(t,\tau)\,\mathrm{e}^{-\lambda\overline{\tau}}\big\|_{\mathbb{X}}\,\big(\|u\|_{\mathcal{U}}\big)^m\,\mathrm{d}\tau\Big) \leq \|h_m\|_{\mathcal{V}^m}\big(\|u\|_{\mathcal{U}}\big)^m$. Hence, the series $\sum_{m\in\mathbb{N}^*}\int_{[0,t]^m} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau$ converges normally in $\mathcal{X}$ to a limit x such that $\|x\|_{\mathcal{X}} = \sup_{t\in\mathbb{R}_+}\Big(\Big\|\sum_{m\in\mathbb{N}^*}\int_{[0,t]^m} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau\Big\|_{\mathbb{X}}\mathrm{e}^{\lambda t}\Big) \leq \sum_{m\in\mathbb{N}^*}\|h_m\|_{\mathcal{V}^m}\big(\|u\|_{\mathcal{U}}\big)^m = \varphi\big(\|u\|_{\mathcal{U}}\big)$. ■

III. MAIN CONVERGENCE RESULTS

In this section, a Volterra series is introduced in definition 4, which converges in a disk with computable radius (theorem 2). The sum is a solution (possibly in a weak sense) of problem (1-5). A bound on the remainders $\|x - x_M\|_{\mathcal{X}}$ is given in theorem 3, where x_M denotes the sum of the Volterra series truncated at order M.

Definition 3 (Index set and selection function): Let $m \in \mathbb{N}^*$ and $J \in \mathbb{N}^*$. The set M_m^J is defined by

$$\mathbb{M}_m^J = \Big\{p \in (\mathbb{N}^*)^J \;\Big|\; p_1 + \ldots + p_J = m\Big\}.$$

Moreover, for all $p \in \mathbb{M}_m^J$ and for all $j \in [1,J]_{\mathbb{N}}$, the selection function $S_p^j : (\mathbb{R}_+)^m \to (\mathbb{R}_+)^{p_j}$ is defined by, denoting $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$,

$$S_p^j(\tau) = \big(\tau_{p_1+\ldots+p_{j-1}+1}, \tau_{p_1+\ldots+p_{j-1}+2}, \ldots, \tau_{p_1+\ldots+p_j}\big).$$

Note that if $J > m$, then $\mathbb{M}_m^J = \emptyset$.

Definition 4 (Kernels recursive construction): Consider system (1-5). Then, the family of kernels $\{h_m\}_{m\in\mathbb{N}^*}$ is defined by $h_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{X}$ with

$$h_1(t,\tau_1) = 1_{\mathbb{R}_+}(t-\tau_1)\,\mathrm{e}^{A(t-\tau_1)}B,$$

and for all $m \geq 2$, $h_m : \mathbb{R}_+ \times (\mathbb{R}_+)^m \to \mathbb{X}$ with

$$h_m(t,\tau) = 1_{\mathbb{R}_+}(t - \max\tau)\left(\int_{\max\tau}^{t} v_m(t,\theta,\tau)\,\mathrm{d}\theta + w_m(t,\tau)\right),$$

where $1_{\mathbb{R}_+}$ denotes the Heaviside function and

$$v_m(t,\theta,\tau) = \mathrm{e}^{A(t-\theta)} \sum_{j=2}^{\min(J,m)} \sum_{p\in\mathbb{M}_m^j} P_j\Big(h_{p_1}\big(\theta, S_p^1(\tau)\big),\dots \\ \dots, h_{p_j}\big(\theta, S_p^j(\tau)\big)\Big), \quad (7)$$

$$w_m(t,\tau) = 1_{\mathbb{R}_+}(\tau_m - \max_{1\le i<m} \tau_i)\,\mathrm{e}^{A\,(t-\tau_m)} \Bigg[\sum_{k=2}^{\min(K,m)} \sum_{\substack{q\in\mathbb{M}_m^k \\ q_k=1}} \\ Q_k\Big(h_{q_1}\big(\tau_m, S_q^1(\tau)\big),\dots, h_{q_{k-1}}\big(\tau_m, S_q^{k-1}(\tau)\big), 1\Big)\Bigg]. \quad (8)$$

This Volterra series provides a formal solution of system (1-5) (see e.g. [4], [6], [15], [16]).

Definition 5 (Function $\mathcal{F}$): The function $\mathcal{F}$ associated to system (1-5) is defined by the rational function

$$\mathcal{F}(X) = \frac{\|h_1\|_{\mathcal{V}^1} + \sum_{k=2}^{K} \mathcal{Q}_k\, X^{k-1}}{1 - \sum_{j=2}^{J} \mathcal{P}_j\, X^{j-1}}, \quad (9)$$

where for all $j \in [2,J]_{\mathbb{N}}$, $k \in [2,K]_{\mathbb{N}}$, $i \in \mathbb{N}^*$,

$$\mathcal{P}_j = \kappa_j \big\|P_j\big\|_{\mathcal{ML}_j(\mathbb{X},\mathbb{X})}, \quad \mathcal{Q}_k = \kappa_k \big\|Q_k\big\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{U},\mathbb{X})}, \quad (10)$$

$$\kappa_i = \sup_{t\in\mathbb{R}_+} \Big(\mathrm{e}^{\lambda t}\int_{[0,t]} \big\|\mathrm{e}^{A(t-\theta)}\big\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \mathrm{e}^{-i\lambda\theta}\mathrm{d}\theta\Big). \quad (11)$$

Remark 2: Since $-a < 0$, this definition of κ_i is consistent and $h_1 \in \mathcal{V}^1$. Indeed, let $\kappa \in \mathbb{R}_+$ be such that for all $\theta \in \mathbb{R}_+$, $\|\mathrm{e}^{A\theta}\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \le \kappa\,\mathrm{e}^{-a\theta}$. Then, $\forall i \in \mathbb{N}^*, 0 < \kappa_i \le \kappa_1 \le \sup_{t\in\mathbb{R}_+}\Big(\mathrm{e}^{\lambda t}\int_0^t \|\mathrm{e}^{A(t-\theta)}\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\,\mathrm{e}^{-\lambda\theta}\mathrm{d}\theta\Big) \le \kappa \sup_{t\in\mathbb{R}_+}\Big(\mathrm{e}^{(\lambda-a)\,t}\int_0^t \mathrm{e}^{(a-\lambda)\theta}\mathrm{d}\theta\Big) = \kappa/(a-\lambda)$, and

$$0 < \|h_1\|_{\mathcal{V}^1} \le \|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}\,\kappa_1 \le \frac{\kappa}{a-\lambda}\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}. \quad (12)$$

Moreover, note that if $\lambda = 0$ then $\kappa_i = \kappa_1 = \int_0^\infty \|\mathrm{e}^{A\theta}\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}\,\mathrm{d}\theta$.

Theorem 2 (Lower bound for the convergence radius): Consider $\lambda \in [0,a[$ and the (non constant) function $\mathcal{F}$ defined by (9). Then, the following results hold:

(i) There exists a unique function $z \mapsto \Psi(z)$, analytic at $z=0$ and such that $\Psi(z) = z\,\mathcal{F}\big(\Psi(z)\big)$. Its radius of convergence is greater than $\rho^\star > 0$ given by

$$\rho^\star = \lim_{x\to+\infty} \frac{x}{\mathcal{F}(x)}, \qquad \text{if } \mathcal{F} \text{ is affine}$$

$$\rho^\star = \frac{\sigma}{\mathcal{F}(\sigma)}, \qquad \text{otherwise,}$$

where σ is the unique solution of $\mathcal{F}(\sigma) - \sigma\,\mathcal{F}'(\sigma) = 0$ on $]0,r[$ where $r \in \mathbb{R}_+^* \cup \{+\infty\}$ is the radius of convergence of $\mathcal{F}$ at $x=0$.

(ii) The family $\{h_m\}_{m\in\mathbb{N}^*}$ in definition 4 belongs to $\mathcal{VS}$ and Ψ is a dominating function of its gain bound function φ on $\mathcal{D}(\rho^\star) = \big\{z\in\mathbb{C}\,\big|\,|z|<\rho^\star\big\}$.

Proof: The proof of (i) is given in lemma 1 in appendix A. The proof of (ii) is divided into two steps: first, a recursive bound on the kernels norm (13) is established proving that $\{h_m\}_{m\in\mathbb{N}^*}$ belongs to $\mathcal{VS}$; second, this bound is used to define the dominating function Ψ and derive its radius of convergence.

Step 1: Following remark 2, h_1 belongs to $\mathcal{V}^1$ and satisfies (12). Moreover, $\forall m\ge 2$, h_m belongs to $\mathcal{V}^m$ and satisfies

$$\|h_m\|_{\mathcal{V}^m} \le \sum_{j=2}^{\min(J,m)} \mathcal{P}_j \sum_{p\in\mathbb{M}_m^j} \prod_{i=1}^{j} \|h_{p_i}\|_{\mathcal{V}^{p_i}} + \\ \sum_{k=2}^{\min(K,m)} \mathcal{Q}_k \sum_{\substack{q\in\mathbb{M}_m^k \\ q_k=1}} \prod_{i=1}^{k-1} \|h_{q_i}\|_{\mathcal{V}^{q_i}}, \quad (13)$$

where $\mathcal{P}_j$ and $\mathcal{Q}_k$ are given in definition 5. Indeed, by induction, let $m\ge 2$ and assume that for $1\le m'\le m-1$, $h_{m'} \in \mathcal{V}^{m'}$ so that $\|h_{m'}\|_{\mathcal{V}^{m'}}$ is finite. For all $t\ge 0$, from definition 4 and recalling the notation $\overline{\tau} = \tau_1+\tau_2+\ldots+\tau_N$,

$$\mathrm{e}^{\lambda t}\int_{[0,t]^m} \Big\| \int_{\max(\tau)}^t v_m(t,\theta,\tau)\,\mathrm{d}\theta + w_m(t,\tau)\Big\|_{\mathbb{X}} \mathrm{e}^{-\lambda\overline{\tau}}\mathrm{d}\tau \\ \le \sum_{j=2}^{\min(J,m)} \sum_{p\in\mathbb{M}_m^j} \mathcal{A}_p(t) + \sum_{k=2}^{\min(K,m)} \sum_{\substack{q\in\mathbb{M}_m^k \\ q_k=1}} \mathcal{B}_q(t), \quad (14)$$

with, from (7-8), for all $p\in\mathbb{M}_m^j$ and $q\in\mathbb{M}_m^k$ such that $q_k=1$,

$$\mathcal{A}_p(t) = \mathrm{e}^{\lambda t}\int_{[0,t]^m}\int_{\max(\tau)}^t \widetilde{\mathcal{A}}_p(t,\theta,\tau)\,\mathrm{d}\theta\;\mathrm{e}^{-\lambda\overline{\tau}}\,\mathrm{d}\tau, \quad (15)$$

$$\widetilde{\mathcal{A}}_p(t,\theta,\tau) = \Big\|\mathrm{e}^{A(t-\theta)}P_j\Big(h_{p_1}\big(\theta,S_p^1(\tau)\big),\dots,h_{p_j}\big(\theta,S_p^j(\tau)\big)\Big)\Big\|_{\mathbb{X}} \quad (16)$$

$$\mathcal{B}_q(t) = \mathrm{e}^{\lambda t}\int_{[0,t]^m} \widetilde{\mathcal{B}}_q(t,\tau)\mathrm{e}^{-\lambda\overline{\tau}}\,\mathrm{d}\tau, \quad (17)$$

$$\widetilde{\mathcal{B}}_q(t,\tau) = \Big\|\mathrm{e}^{A(t-\tau_m)}Q_k\Big(h_{q_1}\big(\tau_m,S_q^1(\tau)\big),\dots \\ \dots, h_{q_{k-1}}\big(\tau_m,S_q^{k-1}(\tau)\big),1\Big)\Big\|_{\mathbb{X}}. \quad (18)$$

Now, on the one hand, for $2\le j\le \min(J,m)$, $p\in\mathbb{M}_m^j$, $t\in\mathbb{R}_+$, $\theta\in[0,t]$, and $\tau\in[0,t]^m$, $\widetilde{\mathcal{A}}_p(t,\theta,\tau) \le \big\|\mathrm{e}^{A(t-\theta)}\big\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \Big\|P_j\Big(h_{p_1}\big(\theta,S_p^1(\tau)\big),\dots,h_{p_j}\big(\theta,S_p^j(\tau)\big)\Big)\Big\|_{\mathbb{X}} \le$

$$\big\|\mathrm{e}^{A(t-\theta)}\big\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\,\big\|P_j\big\|_{\mathcal{ML}_j(\mathbb{X},\mathbb{X})} \prod_{i=1}^{j}\Big\|h_{p_i}\big(\theta,S_p^i(\tau)\big)\Big\|_{\mathbb{X}}. \quad (19)$$

Moreover, for $1\le i\le j$ and $\theta\in[0,t]$,

$$\int_{[0,t]^{p_i}} \big\|h_{p_i}(\theta,\eta)\big\|_{\mathbb{X}} \mathrm{e}^{-\lambda\overline{\eta}}\mathrm{d}\eta \le \int_{\mathbb{R}_+^{p_i}} \|h_{p_i}(\theta,\eta)\|_{\mathbb{X}}\mathrm{e}^{-\lambda\overline{\eta}}\mathrm{d}\eta \le \\ \mathrm{e}^{-\lambda\theta}\Big(\sup_{\theta\in\mathbb{R}_+}\mathrm{e}^{\lambda\theta}\int_{\mathbb{R}_+^{p_i}}\|h_{p_i}(\theta,\eta)\|_{\mathbb{X}}\,\mathrm{e}^{-\lambda\overline{\eta}}\mathrm{d}\eta\Big) \le \mathrm{e}^{-\lambda\theta}\|h_{p_i}\|_{\mathcal{V}^{p_i}}. \quad (20)$$

Hence, from (15-20),

$$\mathcal{A}_p(t) \le \mathcal{P}_j \prod_{i=1}^{j} \|h_{p_i}\|_{\mathcal{V}^{p_i}}, \quad (21)$$

is finite.
On the other hand, for $2 \leq k \leq \min(K,m)$, $q \in \mathbb{M}_m^k$, $t \in \mathbb{R}_+$, and $\tau \in [0,t]^m$, $\widetilde{\mathcal{B}}_q(t,\tau) \leq \left\|\mathrm{e}^{A(t-\tau_m)}\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \times$

$$\left\|Q_k\Big(h_{q_1}\big(\tau_m, S_q^1(\tau)\big),\ldots,h_{q_{k-1}}\big(\tau_m, S_q^{k-1}(\tau)\big),1\Big)\right\|_{\mathbb{X}} \leq$$

$$\left\|\mathrm{e}^{A(t-\tau_m)}\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \left\|Q_k\right\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{U},\mathbb{X})} \prod_{i=1}^{k-1} \left\|h_{q_i}\big(\tau_m, S_q^i(\tau)\big)\right\|_{\mathbb{X}}. \quad (22)$$

Moreover, for all $(i,\tau_m) \in [1,k-1]_{\mathbb{N}} \times [0,t]$, $\int_{[0,t]^{q_i}} \|h_{q_i}(\tau_m,\eta)\|_{\mathbb{X}} \mathrm{e}^{-\lambda\overline{\eta}} \mathrm{d}\eta \leq \int_{\mathbb{R}_+^{q_i}} \|h_{q_i}(\tau_m,\eta)\|_{\mathbb{X}} \mathrm{e}^{-\lambda\overline{\eta}} \mathrm{d}\eta \leq \mathrm{e}^{-\lambda\tau_m}\Big(\sup_{\tau_m\in\mathbb{R}_+} \mathrm{e}^{\lambda\tau_m}\int_{\mathbb{R}_+^{q_i}} \|h_{q_i}(\tau_m,\eta)\|_{\mathbb{X}}\, \mathrm{e}^{-\lambda\overline{\eta}} \mathrm{d}\eta\Big) \leq \mathrm{e}^{-\lambda\tau_m}\|h_{q_i}\|_{\mathcal{V}^{q_i}}$. Hence, from (17-18),

$$\mathcal{B}_q(t) \quad \leq \quad \mathcal{Q}_k \prod_{i=1}^{k-1} \|h_{q_i}\|_{\mathcal{V}^{q_i}}\,, \qquad (23)$$

is finite. Therefore, from (14), (21), and (23), h_m is in $\mathcal{V}^m$ and (13) holds.

Step 2: Consider the formal series $\Psi(X) = \sum_{m\in\mathbb{N}^*} \psi_m X^m$, with $\psi_1 = \|h_1\|_{\mathcal{V}^1}$ and $\forall m \geq 2$, $\psi_m = \sum_{j=2}^{\min(J,m)} \mathcal{P}_j \sum_{p\in\mathbb{M}_m^j} \prod_{i=1}^{j} \psi_{p_i} + \sum_{k=2}^{\min(K,m)} \mathcal{Q}_k \sum_{\left\{\substack{q\in\mathbb{M}_m^k\\ q_k=1}\right.} \prod_{i=1}^{k-1} \psi_{q_i}$.

From (13), it follows by a straightforward induction that

$$\forall m \in \mathbb{N}^*, \quad \|h_m\|_{\mathcal{V}^m} \leq \psi_m. \qquad (24)$$

Moreover, the formal series $\mathcal{R}(X) = \sum_{j=2}^{J} \mathcal{P}_j\big(\Psi(X)\big)^j + X\sum_{k=2}^{K} \mathcal{Q}_k\big(\Psi(X)\big)^{k-1}$ satisfies $\mathcal{R}(X) = \sum_{j=2}^{J}\mathcal{P}_j\Big(\sum_{m\in\mathbb{N}^*}\psi_m X^m\Big)^j + X\sum_{k=2}^{K}\mathcal{Q}_k\Big(\sum_{m\in\mathbb{N}^*}\psi_m X^m\Big)^{k-1} = \sum_{m=2}^{\infty} X^m\Big(\sum_{j=2}^{\min(J,m)}\mathcal{P}_j\sum_{p\in\mathbb{M}_m^j}\prod_{i=1}^{j}\psi_{p_i} + \sum_{k=2}^{\min(K,m)}\mathcal{Q}_k\sum_{\left\{\substack{q\in\mathbb{M}_m^k\\ q_k=1}\right.}\prod_{i=1}^{k-1}\psi_{q_i}\Big) = \sum_{m=2}^{\infty}\psi_m X^m = \Psi(X) - \psi_1 X = \Psi(X) - \|h_1\|_{\mathcal{V}^1} X$.

Then, it follows that $\Psi(X) = X\,\mathcal{F}\big(\Psi(X)\big)$ where $\mathcal{F}$ is defined by (9). Finally, from (i), Ψ is analytic on $\mathcal{D}(\rho^\star)$ and, from (24), Ψ is a dominating function of φ, proving (ii). ∎

Following (i), this theorem makes the numerical computation of σ easy using standard numerical "zero finding" methods, even if $\mathcal{F}$ is a high order rational fraction. Then, the (numerical) computation of $\rho^\star$ is also straightforward.

Theorem 3 (Truncation error bound): Consider system $\mathcal{S}$ (1-5) and assume that $\lambda \in [0,a[$. Let $\mathcal{F}$ be defined as in definition 5, and suppose that $\mathcal{F}$ is not affine. Let σ and $\rho^\star$ be defined as in theorem 2. For all $M \in \mathbb{N}^*$, let $V_M x(t) = \sum_{m=1}^{M}\int_{[0,t]^m} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau$ denote the finite M-order partial sum of the Volterra series. Then, for all $u \in \mathcal{U}$ such that $U = \dfrac{\|u\|_{\mathcal{U}}}{\rho^\star} < 1$, $\left\|x - V_M x\right\|_{\mathcal{X}} \leq \sigma\,\dfrac{U^{M+1}}{1-U}$.

Proof: Let $u \in \mathcal{U}$ be such that $U = \frac{\|u\|_{\mathcal{U}}}{\rho^\star} < 1$. From theorem 2, the gain bound function φ is dominated by Ψ and the Volterra series as well as Ψ are normally convergent on $\mathcal{D}(\rho^\star)$. Hence, denoting $R_M\varphi$, $R_M\Psi$ the remainder at order M of the series φ and Ψ,

$$|R_M\varphi(u)| \leq R_M\varphi(|u|) \leq R_M\Psi(|u|).$$

From lemma 1 in appendix A, Ψ is a positive strictly increasing bijection from $[0,\rho^\star]$ to $[0,\sigma]$ with positive Taylor coefficients at all order, so that on any disk with radius $\rho < \rho^\star$, Cauchy estimates yield $\forall m \in \mathbb{N}^*,\ \psi_m \leq \sigma/\rho^m$. Then, for $\rho \to \rho^\star$, the limit leads to $\psi_m \leq \sigma/(\rho^\star)^m$. Therefore, $\forall M \in \mathbb{N}^*,\ \forall z \in \mathcal{D}(\rho^\star),\ |R_M\Psi(z)| \leq \sigma\,\dfrac{\left(\frac{|z|}{\rho^\star}\right)^{M+1}}{1-\left(\frac{|z|}{\rho^\star}\right)}$. ∎

Remark 3: All the results given in this section remain valid for analytic nonlinear functions P and Q in (4,5). Indeed, the singular inversion theorem can still be applied on $\mathcal{F}$ (see [17]), which is an analytic function in this case.

IV. APPLICATION AND EXAMPLES

In this section, applicative results on two examples are presented in the BIBO case ($\lambda = 0$ so that $\mathcal{U} = \mathcal{B}_{\mathbb{U}}(0)$, $\mathcal{X} = \mathcal{B}_{\mathbb{X}}(0)$) for which analytic computation are given.

A. A 1D system with third order nonlinearity

Let $a \in]0,1[$, $\varepsilon \in \mathbb{R}$, and consider the following system

$$\forall t > 0, \qquad \dot{x} + a\,x - \varepsilon x^3 = u,$$

where u is the input, with zero initial conditions $x(0) = 0$. It corresponds to (1-5) with $\mathbb{U} = \mathbb{R}$ and $\mathbb{X} = \mathbb{R}$, $A = (-a)$, $B = (1)$, $P(x) = \varepsilon x^3$, $Q(x,u) = (0)$. We have $\|h_1\|_{\mathcal{V}^1} = \frac{1}{a}$, and following definition 5, remark 2, and since $\lambda = 0$, it follows that $\mathcal{P}_2 = 0$, $\mathcal{P}_3 = \varepsilon\kappa_3$, $\kappa_3 = \kappa_2 = \kappa_1 = \frac{1}{a}$, and $\mathcal{F}(X) = \dfrac{\|h_1\|_{\mathcal{V}^1}}{1-\varepsilon\kappa_1 X^2} = \dfrac{1}{a-\varepsilon X^2}$. From theorem 2, σ is the positive root of $3\varepsilon X^2 - a$, from which the lower bound $\rho^\star$ of the radius of convergence of the Volterra series is expressed as $\rho^\star = \frac{2}{3}\sqrt{\frac{a^3}{3\varepsilon}}$.

Numerical simulations are performed with $a = 0.65$ and $\varepsilon = 0.04068$ so that $\rho^\star \approx 1$. The input $u(t)$ is constant and equal to e on $[0,\,25]$, and jumps to $-e$ on $[25,\,50]$. Simulations show that $\rho^\star$ is indeed the radius of convergence of the Volterra series: for $e < 1$, it converges to the trajectory of the nonlinear system (see fig. 1ⓐ,ⓑ), whereas for $e > 1$, it becomes divergent and the trajectory of the nonlinear system gets out of the domain of attraction of 0 (see fig. 1ⓒ).

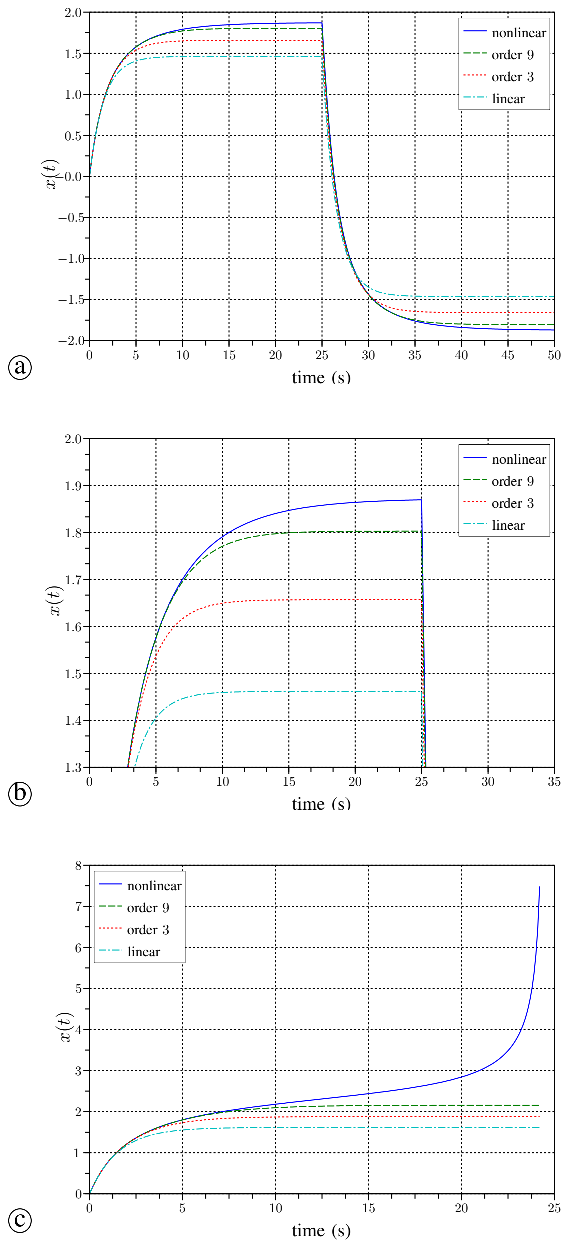

Fig. 1. (example 1) Numerical computation of x for $a = 0.65$, $\varepsilon = 0.04068$ with $e = 0.95 < \rho^\star = 1$ in ⓐ, ⓑ and $e = 1.05 > \rho^\star$ in ⓒ. Figure ⓑ is a zoom of figure ⓐ.

B. A damped Duffing oscillator

Let $a \in]0, 1[$, $\varepsilon \in \mathbb{R}$, and consider the system with input u governed by

$$\forall t > 0, \qquad \ddot{y} + 2a\dot{y} + (1 + \varepsilon y^2)\, y = u, \tag{25}$$

with zero initial conditions $y(0) = 0$, $\dot{y}(0) = 0$. This system defines a damped Duffing oscillator if $\varepsilon > 0$. It takes the form (1-5) where $\mathbb{U} = \mathbb{R}$ and $\mathbb{X} = \mathbb{R}^2$ are associated with the euclidean norm, the state is $x = (y, \dot{y})^T$, the input is u, and $A = \begin{pmatrix} 0 & 1 \\ -1 & -2a \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $P(x) = \begin{pmatrix} 0 \\ -\varepsilon x_1^3 \end{pmatrix}$, $Q(x, u) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so that $\max\Big(\Re\mathrm{e}\big(\mathrm{Spec}\, A\big)\Big) = -a < 0$. Like in sec. IV-A, we obtain $\mathcal{P}_2 = 0$, $\mathcal{P}_3 = \varepsilon\kappa_3$, $\kappa_3 = \kappa_2 = \kappa_1$, and $\mathcal{F}(X) = \dfrac{\|h_1\|_{\mathcal{V}^1}}{1 - \varepsilon\kappa_1 X^2}$. This time, σ is the positive root of $3\varepsilon\kappa_1 X^2 - 1$, from which the lower bound $\rho^\star$ of the radius of convergence of the Volterra series is expressed as $\rho^\star = \frac{2}{3\sqrt{3\kappa_1\varepsilon}\|h_1\|_{\mathcal{V}^1}}$. Numerical simulations were performed with $a = 0.65$ for which $\|h_1\|_{\mathcal{V}^1} \approx 1.693$ and $\kappa_1 \approx 2.496$, and with $\varepsilon = 2.07 \times 10^{-2}$ so that $\rho^\star \approx 1$.

Note that contrarily to IV-A, the uncontrolled system is unconditionally stable and the domain of convergence cannot be related to any basin of attraction boundary. The guaranteed convergence disk given in theorem 2 is a conservative result and numerical simulations suggest that the Volterra series can still converge for some inputs outside the convergence domain (see fig. 2ⓐ). Nevertheless, when the input norm is sufficiently greater than $\rho^\star$, the Volterra series seems to diverge as from the first terms (see fig. 2ⓑ).

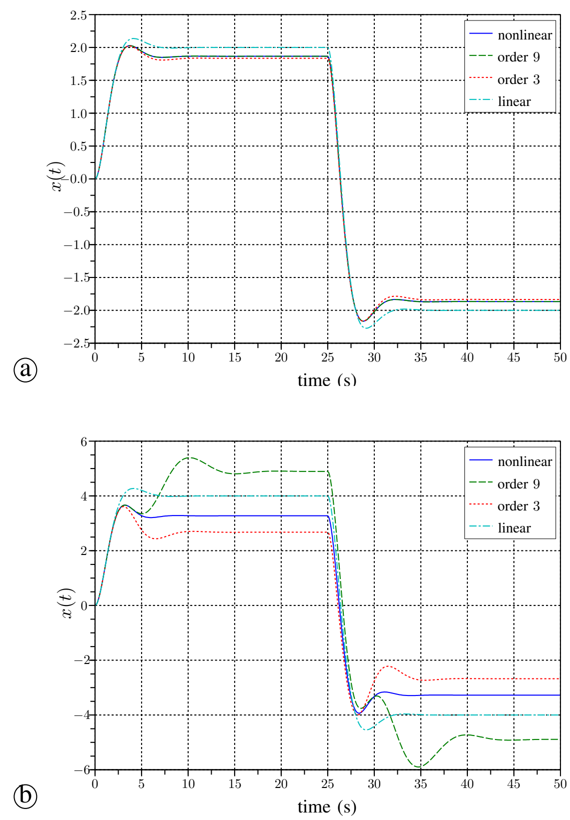

Fig. 2. (example 2) Numerical computation of $y = x_1$ for $a = 0.65$, $\varepsilon = 4.068 \times 10^2$ with the same input shape as in IV-A: $e = 2\rho^\star (= 2)$ in ⓐ and $e = 4\rho^\star$ in ⓑ.

V. Conclusion

Introducing an adequate functional setting, computable bounds of the radius of convergence and of truncation errors of Volterra series expansions have been proposed for fading memory SI systems, polynomial in the state and affine in the input.

The extension of these results to the multiple input case is under study.

A future extension of this work will consist of generalizing the above results to fading memory systems polynomial in the input and to systems with nonzero initial conditions.

Another extension extension will consist of generalizing these results to some classes of infinite dimensional systems. Especially, for boundary and distributed controlled PDE systems solved using Volterra series (see e.g. [18], [19]), no similar convergence results seems to be available to the knowledge of the authors.

VI. ACKNOWLEDGMENTS

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APPENDIX

A. Lemma 1 and proof

Lemma 1: Let $c \in \mathbb{R}_+^*$, $(J, K) \in \mathbb{N}^2 \backslash \{(0,0)\}$ and let $A(X) = \sum_{j=1}^{J} a_j X^j$ and $B(X) = \sum_{k=1}^{K} b_k X^k$ be polynomials with non-negative coefficients. Defining $F(X) = \dfrac{c + B(X)}{1 - A(X)}$, the following results hold:

(i) At $x = 0$, F is nonzero and analytic with nonnegative Taylor coefficients.

(ii) If F is not affine, there exists a unique solution σ of $F(\sigma) - \sigma F'(\sigma) = 0$ on $]0, r[$ where $r \in \mathbb{R}_+^* \cup \{+\infty\}$ is the radius of convergence of F at $x = 0$.

(iii) There exists a unique function $z \mapsto \Psi(z)$, analytic at $z = 0$ and such that $\Psi(z) = z\, F\big(\Psi(z)\big)$. A lower bound of the radius of convergence of this function at $z = 0$ is given by

$$\rho^\star = \lim_{x \to +\infty} \frac{x}{F(x)}, \qquad \text{if } F \text{ is an affine function,}$$
$$\rho^\star = \frac{\sigma}{F(\sigma)}, \qquad \text{otherwise,}$$

where σ is defined in (ii).

Proof: If F is an affine function such that $F(X) = c + b_1 X$ with $b_1 > 0$, then the results (i,iii) are straightforward. The closed-form results are $\Psi(z) = cz/(1 - b_1 z)$ and $\rho^\star = 1/b_1 > 0$.

Now, suppose that the rational function F is not affine and consider the rational fraction H defined by

$$H(X) = \frac{X\, F'(X)}{F(X)}.$$

Then, two cases are considered:
<u>Case 1:</u> If F is polynomial, then (i) is obvious, $r = +\infty$, and $\lim_{x \to r} H(x) = K > 1$ since F is not affine.
<u>Case 2:</u> Otherwise, $A \neq 0$ and the denominator $1 - A$ of F is a strictly decreasing function from 1 to $-\infty$ on $\mathbb{R}_+$ so that, on $\mathbb{R}_+$, it has a unique root $r > 0$. Moreover, for all $z \in \mathbb{C}$ such that $|z| < r$, $|A(z)| \leq A(|z|) < A(r) = 1$ so that $F(z) = \big(c + B(z)\big) \sum_{n=0}^{+\infty} \big(A(z)\big)^n$ is convergent and its Taylor coefficients are finite sums of positive terms, proving (i). On the other hand, as for all $x \in [0, r[$, $H(x) = \dfrac{x\, B'(x)}{c + B(x)} + \dfrac{x\, A'(x)}{1 - A(x)} \geq \dfrac{x\, A'(x)}{1 - A(x)}$, it follows that $\lim_{x \to r^-} H(x) = +\infty$.

Therefore, for both cases $A = 0$ and $A \neq 0$, (i) is true and $\lim_{x \to r^-} H(x) > 1$. Finally, the hypotheses of the singular inversion theorem (see e.g proposition IV.5 and theorem VI.6 in [17]) are met, and its application proves (ii) and (iii). ∎

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