Convergence of series expansions for some infinite dimensional nonlinear systems

Thomas Hélie * **Béatrice Laroche** **

* *Ircam - CNRS STMS UMR 9912, 1 place Igor Stravinsky, 75004 Paris, France (e-mail: thomas.helie@ircam.fr).*
** *Laboratoire des Signaux et Systèmes, Université Paris-Sud, CNRS UMR 8506, Supélec, 91405 Gif-Sur-Yvette, France (e-mail: beatrice.laroche@lss.supelec.fr).*

Abstract
Volterra series expansions have been extensively used to solve and represent the dynamics of weakly nonlinear finite dimensional systems. Such expansions can be recovered by using the regular perturbation method and choosing the input of the system as the perturbation: the state (or the output) is then described by a series expansion composed of homogeneous contributions with respect to the input, from which kernels of convolution type can be deduced. This paper provides an extension (based on this approach) to a class of semilinear infinite dimensional systems, nonlinear in state and affine in input. As a main result, computable bounds of the convergence radius of the series are established. They characterize domains on which the series defines a mild solution of the system. The convergence criterion is established for bounded signals (infinite norms on finite or infinite time intervals) as follows: first, norm estimates of the series expansion terms are derived; second, the singular inversion theorem is used to deduce an easily computable bound of the convergence radius. In the formalism proposed here, non zero initial conditions can be also considered as a perturbation so that no precomputation of nominal trajectories is required in practice. The relevance of the method is illustrated on an academic example.

Keywords: Nonlinear systems, perturbation analysis, partial differential equations, convergence proofs

1. INTRODUCTION

Volterra series is a functional series expansion of the solution of nonlinear controlled systems, first introduced by the Italian mathematician Volterra (see Volterra (1959)). This tool has been extensively used in signal processing and control, electronics and electro-magnetic waves, mechanics and acoustics, bio-medical engineering, for modeling, identification and simulation purposes. There exists a vast literature concerning Volterra series. Among others, they were studied by Brockett (1976); Gilbert (1977); Fliess et al. (1983); Isidori (1995) from the geometric control point of view, and in Rugh (1981); Crouch and Collingwood (1987); Schetzen (1989) from the input-output representation and realization point of view. Most of applications address finite dimensional systems even if a few cases of some infinite dimensional problems has been investigated (see e.g. Hélie and Hasler (2004); Hélie and Roze (2008)).

Only a few results address the convergence of such series expansions. The existence of a non zero convergence radius for complex linear analytic finite dimensional systems with no initial conditions has been proved by Brockett (1977). Other theoretical and local-in-time results are known (see e.g. Isidori (1995); Lamnabhi-Lagarrigue (1994)). Results on fading memory have be investigated by Boyd and Chua (1985). More recently, results have been obtained in the frequency domain by Jing et al. (2008); Peng and Lang (2007), results relying on regular perturbations are given by Bullo (2002). We have also established computable bounds of the convergence radius for finite dimensional systems with a polynomial nonlinearity (see Hélie and Laroche (2008, 2009)).

This paper focuses on the computation of guaranteed convergence bounds of a series expansion which extends the Volterra series formalism to the case of a class of semilinear infinite dimensional systems, nonlinear in state and affine in input. The convergence criterion is established for bounded signals ($L^\infty(\mathbb{R}_+)$ or $L^\infty([0,T])$ norms, with $T>0$). The derivation of the result proceeds in two steps. First, norm estimates of the series expansion terms are derived. Second, the singular inversion theorem is used to deduce an easily computable bound of the convergence radius. Moreover, in the formalism proposed here, non zero initial conditions can be also considered as a perturbation so that no precomputation of nominal trajectories is required in practice.

The paper is organized as follows: section 2 defines the notations, the functional setting, the class of systems under consideration and recalls some general definitions and basic properties of Volterra series. Section 3 introduces the series expansion which provides a solution of the system state and establishes the convergence results. These results are illustrated on an academic example in section 4. Finally, conclusions and perspectives are given in section 5.

2. GENERAL FRAMEWORK

2.1 Notations and functional setting

The following notations and functional setting are introduced:

- $\mathbb{T}$ denotes the time interval $[0,T]$ with $T>0$ or $\mathbb{R}_+$.
- $\mathbb{U}$ and $\mathbb{X}$ are Banach spaces on the field $\mathbb{R}$.
- $\mathcal{L}(\mathbb{U},\mathbb{X})$ and $\mathcal{L}(\mathbb{X})$ are the sets of bounded linear operators from $\mathbb{U}$ to $\mathbb{X}$, and from $\mathbb{X}$ to $\mathbb{X}$, respectively.
- $\mathcal{ML}_j(\mathbb{X},\mathbb{X})$ ($j\geq 2$) is the set of bounded multilinear operators from $\underbrace{\mathbb{X}\times\cdots\times\mathbb{X}}_{j}$ to $\mathbb{X}$, with norm
$$\|E\| = \sup_{\substack{(x_1,\ldots,x_j)\in\mathbb{X}^j\\ \|x_1\|=\cdots=\|x_j\|=1}} \|E(x_1,\ldots,x_j)\|.$$
- $\mathcal{ML}_{j,k}(\mathbb{X},\mathbb{U},\mathbb{X})$ ($j\geq 1$, $k\geq 1$) is the set of bounded multilinear operators from $\underbrace{\mathbb{X}\times\cdots\times\mathbb{X}}_{j}\times\underbrace{\mathbb{U}\times\cdots\times\mathbb{U}}_{k}$ to $\mathbb{X}$, with norm
$$\|E\| = \sup_{\substack{(x_1,\ldots,x_j,u_1,\ldots,u_k)\in\mathbb{X}^j\times\mathbb{U}^k\\ \|x_1\|=\cdots=\|u_k\|=1}} \|E(x_1,\ldots,x_j,u_1,\ldots,u_k)\|.$$
- $\mathcal{U} = L^\infty(\mathbb{T},\mathbb{U})$ and $\mathcal{X} = L^\infty(\mathbb{T},\mathbb{X})$ are standard Lebesgue spaces.

2.2 System under consideration

Consider the class of infinite-dimensional nonlinear causal system governed by

$$\dot{x} = A\,x + B\,u + P(x) + Q(x,u), \text{ for } t\in\mathbb{T}, \quad (1)$$
$$x(0) = x_{\text{ini}} \in \mathbb{X}. \quad (2)$$

Operator A is closed and generates a strongly continuous semigroup S on $\mathbb{X}$ with growth bound α which is assumed to be strictly negative if $\mathbb{T}=\mathbb{R}_+$. Moreover, $\beta>0$ denotes the lowest constant such that for all $t\in\mathbb{T}$, $\|S(t)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\leq\beta\exp(\alpha t)$. Operator B belongs to $\mathcal{L}(\mathbb{U},\mathbb{X})$. Moreover,

$$P(x) = \sum_{k=2}^{+\infty} A_k(\underbrace{x,\ldots,x}_{k}), \quad (3)$$

$$Q(x,u) = \sum_{k=2}^{+\infty} B_k(\underbrace{x,\ldots,x}_{k-1},u), \quad (4)$$

where $A_k\in\mathcal{ML}_j(\mathbb{X},\mathbb{X})$ and $B_k\in\mathcal{ML}_{k-1,1}\big(\mathbb{X},\mathbb{U},\mathbb{X}\big)$ are multilinear bounded operator such that $\sum_{k=2}^{+\infty}\|A_k\|_{\mathcal{ML}_k(\mathbb{X},\mathbb{X})}z^k$ and $\sum_{k=2}^{+\infty}\|B_k\|_{\mathcal{ML}_{k,1}(\mathbb{X},\mathbb{U},\mathbb{X})}\,z^{k-1}$ are analytic at $z=0$.

Definition 1. (Mild solution). Let $u\in\mathcal{L}^\infty_{loc}(\mathbb{T},\mathbb{U})$. Then, x is said to be a *mild solution* of (1-4) iff $x\in\mathcal{C}_0(\mathbb{T},\mathbb{X})$ and satisfies, $\forall t\in\mathbb{T}$,

$$x(t) = S(t)x_{\text{ini}} + \int_0^t S(t-\tau)\Big(Bu(\tau) + P\big(x(\tau)\big) + Q\big(x(\tau),u(\tau)\big)\Big)\,\mathrm{d}\tau.$$

3. REGULAR PERTURBATION METHOD: COMPUTABLE CONVERGENCE RESULTS

We look for an expansion of the trajectories of system (1,2) using a regular perturbation approach, where the input u and the initial condition x_{ini} are considered as perturbations. Setting $u=\eta\tilde{u}$ and $x_{\text{ini}}=\eta\tilde{x_{\text{ini}}}$, we look for a solution under the form
$$x = \sum_{m=0}^{\infty}\eta^m\tilde{x}_m = \sum_{m=0}^{\infty}x_m, \text{ with } x_m=\eta^m\tilde{x}_m.$$
Replacing x in (1,2) and sorting along the powers of η yield $x_0=0$ and the following formal expressions, for all $t\in\mathbb{T}$,

$$x_1(t) = S(t)x_{\text{ini}} + \int_0^t S(t-\tau)\,B\,u(\tau)\,\mathrm{d}\tau, \quad (5)$$

$$x_m(t) = \int_0^t S(t-\tau)\chi_m(\tau)\,\mathrm{d}\tau, \qquad \text{for } m\geq 2, \quad (6)$$

where

$$\chi_m(\tau) = \sum_{k=2}^{m}\sum_{p\in\mathbb{M}_m^k} A_k\Big(x_{p_1}(\tau),\ldots,x_{p_k}(\tau)\Big) + \sum_{k=2}^{m}\sum_{\begin{cases}q\in\mathbb{M}_m^k\\ q_k=1\end{cases}} B_k(x_{q_1}(\tau),\ldots,x_{q_{k-1}}(\tau),u(\tau)\Big),$$

and the multiple index set $\mathbb{M}_m^K$ is defined for all $m\in\mathbb{N}^*$ and $K\in\mathbb{N}^*$ by
$$\mathbb{M}_m^K = \Big\{p\in(\mathbb{N}^*)^K \;\Big|\; p_1+\cdots+p_K=m\Big\}.$$
As a standard result (see e.g. Pazy (1983)), (5) defines a mild solution $x_1\in\mathcal{C}_0(\mathbb{T},\mathbb{X})$ of the linearized problem. Moreover, by induction, for all $m\in\mathbb{N}^*$, $x_m\in\mathcal{C}_0(\mathbb{T},\mathbb{X})$.

This expansion provides a generalization to the infinite dimensional case of the standard Volterra series expansions (when $x_{\text{ini}}=0$) defined for finite dimensional nonlinear systems (see e.g. Volterra (1959); Rugh (1981); Boyd et al. (1984)).

Lemma 2. Let $u\in\mathcal{U}$. Then, $x_1\in\mathcal{X}$ and $\|x_1\|_{\mathcal{X}}\leq\kappa_1\|x_{\text{ini}}\|_{\mathbb{X}}+\kappa_2\|u\|_{\mathcal{U}}$ where

$$\kappa_1=\beta\max(1,\mathrm{e}^{\alpha T}),\quad \kappa_2=\beta\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}\frac{\mathrm{e}^{\alpha T}-1}{\alpha},\quad \text{if } \mathbb{T}=[0,T],$$

(κ_2 degenerates into $\beta T\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}$ when $\alpha=0$), and where $\kappa_1=\beta$, $\kappa_2=\beta\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}/|\alpha|$, if $\mathbb{T}=\mathbb{R}_+$.

The proof is a standard.

Lemma 3. For all $m\in\mathbb{N}^*$, χ_m and x_m belong to $\mathcal{X}$. Moreover, for all $m\geq 2$,

$$\|x_m\|_{\mathcal{X}} \leq \sum_{k=2}^{m}\Bigg[a_k\sum_{p\in\mathbb{M}_m^k}\prod_{i=1}^{k}\|x_{p_i}\|_{\mathcal{X}} + b_k\sum_{\begin{cases}q\in\mathbb{M}_m^k\\ q_k=1\end{cases}}\Bigg(\prod_{i=1}^{k-1}\|x_{q_i}\|_{\mathcal{X}}\Bigg)\|u\|_{\mathcal{U}}\Bigg], \quad (7)$$

where $a_k = \gamma \left\|A_k\right\|_{\mathcal{ML}_k(\mathbb{X},\mathbb{X})}$, $b_k = \gamma \left\|B_k\right\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{U},\mathbb{X})}$ and with

$$\int_{\mathbb{T}} \left\|S(\theta)\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \mathrm{d}\theta \leq \gamma < \infty.$$

The best estimate in (7) is obtained for $\gamma = \int_{\mathbb{T}} \left\|S(\theta)\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \mathrm{d}\theta$.

Proof. From lemma 2, $x_1 \in \mathcal{X}$. Now, by induction, let $m \geq 2$ and assume that for $1 \leq m' \leq m-1$, $x_{m'} \in \mathcal{X}$. For all $\tau \in \mathbb{T}$,

$$\begin{aligned}
\|\chi_m(\tau)\|_{\mathbb{X}} \leq& \sum_{k=2}^{m} \Bigg[\sum_{p \in \mathbb{M}_m^k} \|A_k\|_{\mathcal{ML}_k} \prod_{i=1}^{k} \|x_{p_i}(\tau)\|_{\mathbb{X}} \\
&+ \sum_{\substack{q \in \mathbb{M}_m^k \\ q_k = 1}} \|B_k\|_{\mathcal{ML}_{k-1,1}} \left(\prod_{i=1}^{k-1} \|x_{q_i}(\tau)\|_{\mathbb{X}} \right) \|u(\tau)\|_{\mathbb{U}} \Bigg] \\
\leq& \sum_{k=2}^{m} \Bigg[\|A_k\|_{\mathcal{ML}_k} \sum_{p \in \mathbb{M}_m^k} \prod_{i=1}^{k} \|x_{p_i}\|_{\mathcal{X}} \\
&+ \|B_k\|_{\mathcal{ML}_{k-1,1}} \sum_{\substack{q \in \mathbb{M}_m^k \\ q_k = 1}} \left(\prod_{i=1}^{k-1} \|x_{q_i}\|_{\mathcal{X}} \right) \|u\|_{\mathcal{U}} \Bigg].
\end{aligned}$$

It follows that $\chi_m \in \mathcal{X}$ and that, for all $t \geq 0$,

$$\begin{aligned}
\|x_m(t)\|_{\mathbb{X}} \leq& \int_0^t \|S(t-\tau)\|_{\mathcal{L}} \, \|\chi_m(\tau)\|_{\mathbb{X}} \, \mathrm{d}\tau \\
\leq& \gamma \sum_{k=2}^{m} \Bigg[\|A_k\|_{\mathcal{ML}_k} \sum_{p \in \mathbb{M}_m^k} \prod_{i=1}^{k} \|x_{p_i}\|_{\mathcal{X}} \\
&+ \|B_k\|_{\mathcal{ML}_{k-1,1}} \sum_{\substack{q \in \mathbb{M}_m^k \\ q_k = 1}} \left(\prod_{i=1}^{k-1} \|x_{q_i}\|_{\mathcal{X}} \right) \|u\|_{\mathcal{U}} \Bigg],
\end{aligned}$$

which proves that $x_m \in \mathcal{X}$ and that (7) holds.

Let $G \in \mathcal{L}(\mathbb{X}, \mathcal{X})$ and $H \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ be defined by

$$\begin{aligned}
G : x_{\mathrm{ini}} &\longmapsto \big(t \mapsto S(t) x_{\mathrm{ini}}\big) \\
H : u \quad &\longmapsto \left(t \mapsto \int_0^t S(t-\tau)\, B\, u(\tau)\, \mathrm{d}\tau\right).
\end{aligned}$$

Then, (5) rewrites

$$x_1 = G\, x_{\mathrm{ini}} + H\, x_1, \tag{8}$$

and we have the following theorem (main result of the paper).

Theorem 4. (Convergence criterion). Let $\omega \geq \|H\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} > 0$. Then, for all $\varepsilon \in [0, 1]$:

(i) Consider the analytic functions

$$a(z) = \sum_{k=2}^{+\infty} a_k z^{k-1}, \quad b(z) = \sum_{k=2}^{+\infty} b_k z^{k-1},$$

where a_k and b_k are defined in lemma 3, and define the (non constant) function

$$F_\varepsilon(z) = \frac{\omega + \varepsilon b(z)}{1 - a(z)},$$

with convergence radius $r \in \mathbb{R}_+^* \cup \{+\infty\}$ at $z = 0$. Equation $x\, F_\varepsilon'(x) - F_\varepsilon(x) = 0$ has either one solution denoted σ (case 1) or zero solution (case 2), in $]0, r[$. Let $\rho_\varepsilon^\star > 0$ be defined by

$$\text{(case 1)} \qquad \rho_\varepsilon^\star = \frac{\sigma}{F(\sigma)}, \tag{9}$$

$$\text{(case 2)} \qquad \rho_\varepsilon^\star = \lim_{x \to r^-} \frac{x}{F(x)}. \tag{10}$$

There exists a unique function $z \mapsto \Phi_\varepsilon(z)$ analytic at $z = 0$ and such that

$$\Phi_\varepsilon(z) = z\, F_\varepsilon\big(\Phi_\varepsilon(z)\big).$$

Its convergence radius is equal to (case 1) or greater than (case 2) $\rho_\varepsilon^\star$.

(ii) Let $u \in \mathcal{U}$ and $x_{\mathrm{ini}} \in \mathbb{X}$ be such that

$$(1-\varepsilon)\|u\|_{\mathcal{U}} \leq \frac{\varepsilon}{\omega} \, \|G(x_{\mathrm{ini}})\|_{\mathcal{X}}, \tag{11}$$

$$\|u\|_{\mathcal{U}} + \frac{1}{\omega} \, \|G(x_{\mathrm{ini}})\|_{\mathcal{X}} < \rho_\varepsilon. \tag{12}$$

Then, the series $x = \sum_{m \in \mathbb{N}^*} x_m$ defined from (5-6) is normally convergent in $\mathcal{X}$ and

$$\left\|x\right\|_{\mathcal{X}} \leq \Phi_\varepsilon\Big(\|u\|_{\mathcal{U}} + \frac{1}{\omega} \, \|G(x_{\mathrm{ini}})\|_{\mathcal{X}}\Big).$$

More precisely, $z \mapsto \Phi_\varepsilon\Big(z\big(\|u\|_{\mathcal{U}} + \frac{1}{\omega} \, \|G(x_{\mathrm{ini}})\|_{\mathcal{X}}\big)\Big)$ is a dominating function of $z \mapsto \sum_{m \in \mathbb{N}^*} \|x_m\|_{\mathcal{X}} z^m$ for $|z| < 1$.

Remark 5. If $b = 0$, then F_ε, σ_ϵ, $\rho_\varepsilon^\star$ and (12) do not depend on ε. Moreover, in this case, (11) is no longer a constraint since there exists $\varepsilon \in [0, 1]$ such that (11) is satisfied.

Remark 6. If $x_{\mathrm{ini}} = 0$ (zero initial conditions), choosing $\varepsilon = 1$ makes condition (11) trivial and the convergence condition reduces to

$$\|u\|_{\mathcal{U}} < \rho_1.$$

In this case, parameter ρ_1 can be interpreted as a convergence radius.

Remark 7. If $u = 0$ (uncontrolled), choosing $\varepsilon = 0$ makes condition (11) trivial so that the convergence condition reduces to

$$\|G(x_{\mathrm{ini}})\|_{\mathcal{X}} < \omega\, \rho_0.$$

Note that in this case, F_0 and ρ_0 do not involve function b (related to Q in (4)).

Proof. The proof of (i) is a straightforward consequence of lemma 9 in appendix A.

Now for (ii), consider the problem (1,2) with input u and initial condition x_{ini}. Define $\phi_1 = \omega$ and, for $m \geq 2$,

$$\phi_m = \sum_{k=2}^{m} a_k \sum_{p \in \mathbb{M}_m^k} \prod_{i=1}^{k} \phi_{p_i} + \varepsilon \sum_{k=2}^{m} b_k \sum_{\substack{q \in \mathbb{M}_m^k \\ q_k = 1}} \prod_{i=1}^{k-1} \phi_{q_i}.$$

From (8), it follows that

$$\begin{aligned}
\|x_1\|_{\mathcal{X}} &\leq \|H\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} \|u\|_{\mathcal{U}} + \|G(x_{\mathrm{ini}})\|_{\mathcal{X}} \\
&\leq \omega \|u\|_{\mathcal{U}} + \|G(x_{\mathrm{ini}})\|_{\mathcal{X}} \\
&\leq \phi_1\big(\|u\|_{\mathcal{U}} + \frac{1}{\omega} \, \|G(x_{\mathrm{ini}})\|_{\mathcal{X}}\big).
\end{aligned}$$

Moreover, from (11), $\|u\| \leq \varepsilon\big(\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}}\big)$ so that, from lemma 3 and by induction,

$$\forall m \geq 2, \quad \|x_m\|_{\mathcal{X}} \leq \phi_m\big(\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}}\big)^m. \quad (13)$$

Consider the function $\widetilde{a}(X) = X\,a(X)$ and the formal series $\Phi(X) = \sum_{n\in\mathbb{N}^*} \phi_m X^m$. Then, Φ satisfies

$$\begin{aligned}
&\widetilde{a}\big(\Phi(X)\big) + \varepsilon X\, b\big(\Phi(X)\big)\\
&= \sum_{k=1}^{+\infty} a_k\big(\Phi(X)\big)^k + \varepsilon X \sum_{k=1}^{+\infty} b_k\big(\Phi(X)\big)^{k-1}\\
&= \sum_{m=2}^{+\infty} X^m\Bigg[\sum_{k=2}^{m} a_k \sum_{p\in\mathbb{M}_m^k} \prod_{i=1}^{k} \phi_{p_i} + \varepsilon \sum_{k=2}^{m} b_k \sum_{\begin{cases} q\in\mathbb{M}_m^k \\ q_k = 1\end{cases}} \prod_{i=1}^{k-1} \phi_{q_i}\Bigg]\\
&= \sum_{m=2}^{+\infty} \phi_m X^m = \Phi(X) - \phi_1 X = \Phi(X) - \omega X.
\end{aligned}$$

This leads to $X\Big(\omega + \varepsilon b\big(\Phi(X)\big)\Big) = \Phi(X)\,\Big(1 - a\big(\Phi(X)\big)\Big)$, which rewrites $\Phi(X) = X\,F_\varepsilon\big(\Phi(X)\big)$.

Finally, from (i), Φ is analytic on the open disk with radius ρ_ε so that, if $\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}} < \rho_\varepsilon$, then the positive series $\sum_{m\in\mathbb{N}^*} \phi_m\,\Big(\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}}\Big)^m$ converges, $\sum_{m\in\mathbb{N}^*} x_m$ is normally convergent and is bounded by $\Phi\Big(\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}}\Big)$. This concludes the proof.

Remark 8. (Algorithm). Following theorem 4, the convergence parameter ρ_ε can be computed either numerically or analytically, using this algorithm:

Step 1: Compute exact or overestimated values of $\omega = \|H\|_{\mathcal{L}(\mathcal{U},\mathcal{X})}$ (see (8)) and $\gamma = \int_{\mathbb{T}} \big\|S(\theta)\big\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\,\mathrm{d}\theta$ related to the semi-group of the linearized system defined by (1,2) with $P = 0$, $Q = 0$.
Step 2: Compute a_k, b_k and derive F_ε.
Step 3: Compute the unique positive solution σ_ε of the equation $F_\varepsilon(\sigma) - \sigma\,F_\varepsilon'(\sigma) = 0$ if any.
Step 4 Compute $\rho_\varepsilon^\star$ using (9-10).

Note that, in step 1, overestimated values can be easily derived from the growth bound α, parameter β and $\|B\|_{\mathcal{L}(\mathbb{U},\mathbb{X})}$ (as in lemma 2).

It can be easily checked that, when the convergence condition is satisfied, the series $\sum_{m=0}^{\infty} x_m$ defines a mild solution of (1-2) in the sense of definition 1.

4. EXAMPLE

We illustrate our method on a simple academic example.

4.1 System under consideration

Consider the 1D reaction-diffusion process with Dirichlet boundary conditions, described by

$$\partial_t f(t,z) = \nu\partial_z^2 f(t,z) - \mu f(t,z) + \big[f(t,z)\big]^2 + h(z)u(t), \quad (14)$$

$$f(t,0) = f(t,1) = 0, \quad (15)$$

$$f(0,z) = 0 \quad (16)$$

where f is defined on $\mathbb{T}\times[0,1]$ with $\mathbb{T} = \mathbb{R}_+$, h belongs to $C\big([0,1]\big)$ and ν, μ are positive constants.

These equations take the form (1-2) where $\mathbb{U} = \mathbb{R}$, $\mathbb{X} = C_0\big([0,1]\big)$ (space of continuous functions on $[0,1]$ that vanish on the boundary, equipped with the supremum norm). Operator A is defined by

$$A = \nu\partial_z^2 - \mu\, I,$$

with domain $\mathcal{D}(A) = H^2(0,1) \cap H_0^1(0,1)$. Operator B is defined by

$$B(u) : z \mapsto h(z)u.$$

Moreover, $P(x) = A_2(x,x)$ with $A_2 : (x,y) \mapsto xy$ and $Q = 0$.

4.2 Computation of $\rho^\star$

We follow the algorithm steps given in remark 8.

Step 1 (parameters ω and γ) A straightforward computation shows that the eigenvalues of A are $\lambda_n = -(\mu + \nu(n\pi)^2)$ for $n \in \mathbb{N}^\star$. The first eigenvalue of A is therefore $\lambda_1 = -(\mu + \nu\pi^2)$.

Following Cazenave and Haraux (1998), A is the infinitesimal generator of a semigroup of contraction S whose growth bound is $\alpha = \lambda_1$, and the following estimate hold

$$\|S(t)\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \leq M\, \exp(\lambda_1 t),$$

with $M = \exp\left(\frac{\pi}{4}\right)$. We therefore have

$$\int_{\mathbb{T}} \big\|S(\theta)\big\|_{\mathcal{L}(\mathbb{X},\mathbb{X})}\mathrm{d}\theta \leq M\,\frac{1 - \exp(\lambda_1 T)}{|\lambda_1|},$$

so that we can choose the overestimated value

$$\gamma = M\,\frac{1 - \exp(\lambda_1 T)}{|\lambda_1|}.$$

It may happen in some cases (i.e. for some particular functions $z \mapsto h(z)$) that a better value for M is available.

In this example, we use the tight bound

$$\omega = \|H\|_{\mathcal{L}(\mathcal{U},\mathcal{X})} = \sup_{t\in\mathbb{T}} \int_0^t \|S(t-\tau)\,h\|_{\mathbb{X}}\,\mathrm{d}\tau.$$

Step 2 (parameters a_k, b_k and function F_ε) Operator A_2 is such that $\|A_2\|_{\mathcal{ML}_2(\mathbb{X},\mathbb{X})} = 1$, so that $a_2 = \gamma$ and that

$$F_\varepsilon(X) = F(X) = \frac{\omega}{1 - \gamma X},$$

which does note depend on ε as stated in remark 5.

Step 3 (solution σ) Solving $F(\sigma) - \sigma\,F'(\sigma) = 0$ yields

$$\sigma = \frac{1}{2\gamma}.$$

Step 4 (solution $\rho^\star$) $\rho^\star = \dfrac{1}{4\omega\gamma}.$

Convergence criterion Following remark 5, the convergence condition is given by

$$\|u\|_{\mathcal{U}} + \frac{1}{\omega}\,\|G(x_{\text{ini}})\|_{\mathcal{X}} < \rho^\star.$$

Convergence parameter $\rho^\star$ depends on the time interval $\mathbb{T}$ (involved in $\|.\|_{\mathcal{X}}$ and $\|.\|_{\mathcal{U}}$). Figure 1 exhibits $T \mapsto \rho^\star$ for finite time intervals $\mathbb{T} = [0, T]$.

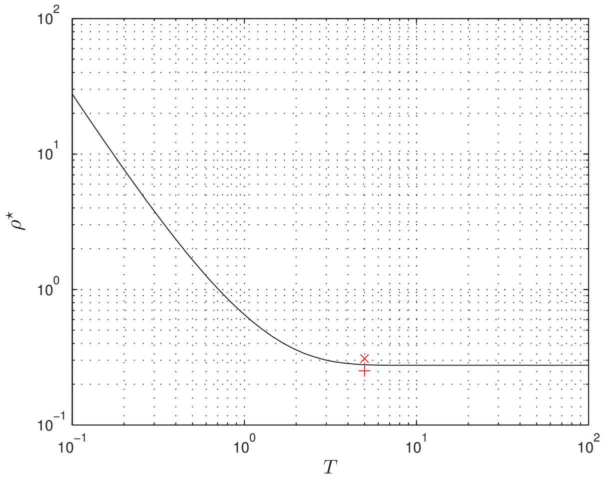

Figure 1. Function $T \mapsto \rho^\star$ computed for $T \in [10^{-1}, 10^2]$ and parameters $\nu = 0.005$, $\mu = 1$ and $h : z \mapsto sin(\pi z)$. Markers $\times$ and $+$ correspond to choices made for simulations (see section 4.3).

4.3 Numerical simulations

Numerical simulations are performed for $\nu = 0.005$, $\mu = 1$ and $h : z \mapsto sin(\pi z)$ (first eigenfunction of A). In this case, it is easy to find that $M = 1$ is the bound for γ. The discretization steps used for the simulations are $\delta z = 0.02$ for the space and $\delta t = 0.1$ for the time. The convergence is studied on $\mathbb{T} = [0, T]$ with $T = 5$ (corresponding to $\rho^\star \approx 0.2783$) and simulations are computed on a longer duration (40 seconds) in order to reveal the behaviour beyond the guanranteed limits.

The first simulation is performed for $x_{\text{ini}} = 0$ and the constant input $u(t) = 0.9\rho^\star$ for $t \geq 0$. Figure 2 displays the reference in ⓐ and the deviation between this reference and the solution computed using the expansion $\sum_{m\geq 1} x_m(t)$ truncated at order 6 with (5-6). The deviation is globally lower than 0.01 (that is less than 3%) on $t \in [0, 40]$ and much lower than 0.001 on $t \in [0, 5]$ on which the convergence is guaranteed.

A comparison between the reference, the expansion truncated at oder 6, and that truncated at order 1 (linearized system) is detailed in figure 3ⓐ for the signal $f(t, 0.5)$. Figures 3ⓑ and 3ⓒ correspond to cases which satisfy the convergence criterion in the same way ($\|u\|_{\mathcal{U}} + \frac{1}{\omega}\|G(x_{\text{ini}})\|_{\mathcal{X}} = 0.9\rho^\star$) but with non zero initial conditions. More precisely, in figure 3ⓒ, $u = 0$ and $\frac{1}{\omega}\|G(x_{\text{ini}})\|_{\mathcal{X}} = 0.9\rho^\star$ where $x_{\text{ini}}(z)$ is zero if $|z - 1/2| > 1/8$ and a (non zero) constant otherwise. In figure 3ⓒ, the contribution of the input and of the initial condition are equally balanced in the convergence criterion, namely, $\omega\|u\|_{\mathcal{U}}/\|G(x_{\text{ini}})\|_{\mathcal{X}} = 1$. As expected, in all these cases, the accuracy of expansions truncated at order 6 is fair at least for $t < 5$.

Figure 4 displays the results obtained with $x_{\text{ini}} = 0$ and $u(t) = 1.11\rho^\star$ (corresponding to marker $\times$ in figure 1). Notice that parameter $1.11\rho^\star$ for $T = 5$ corresponds to the

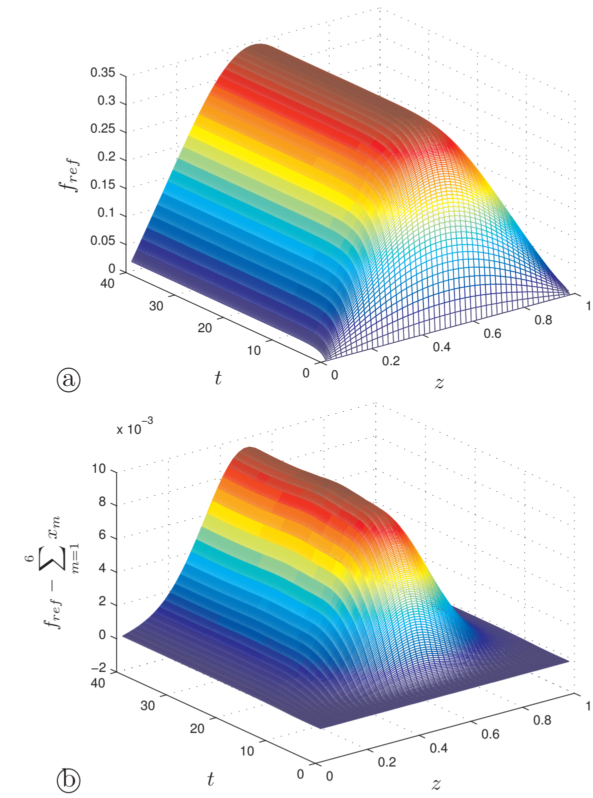

Figure 2. Simulation for $x_{\text{ini}} = 0$ and $u(t) = 0.9\rho^\star$: ⓐ is the reference which is computed using a standard numerical version of A and a standard solver (`ode15s` in Matlab), ⓑ is the deviation between the reference and the solution (5-6) truncated at order 6. These simulations are associated to marker $+$ in figure 1.

convergence parameter $\rho^\star$ for $T \approx 3$ (see figure 1). Hence, the convergence of the expansion is guaranteed until $3\,s$. This is actually in accordance with the results which are plotted in figure 4. Moreover, for this input, the system is no more stable, as exhibited by the reference. This shows that, on this example, $\rho^\star$ gives an accurate bound of the convergence in $\mathcal{X}$ (even if it does not always lead to optimal bounds since inequality (13) is not guaranteed to be optimal).

5. CONCLUSION

Computable convergence bounds of a series expansion which yields exact mild solutions have been established, for a class of infinite dimensional systems that are nonlinear in state and affine in input. This series can be interpreted as an extension of Volterra series. Our results bring a useful contribution in all the applications where series expansion with a guaranteed precision are needed (e.g. simulation and model order reduction). It can also be used for the characterization of stability domains of nonlinear systems (using the $L^\infty(\mathbb{R}_+)$ norm), as well as e.g. the optimization of parameterized stabilizing controllers through the maximization of the convergence parameter ρ_ε.

The extension of these results to the multiple input case is under study. In the near future we also plan to generalize the above results to systems that are, in addition to the above assumptions, nonlinear in input.

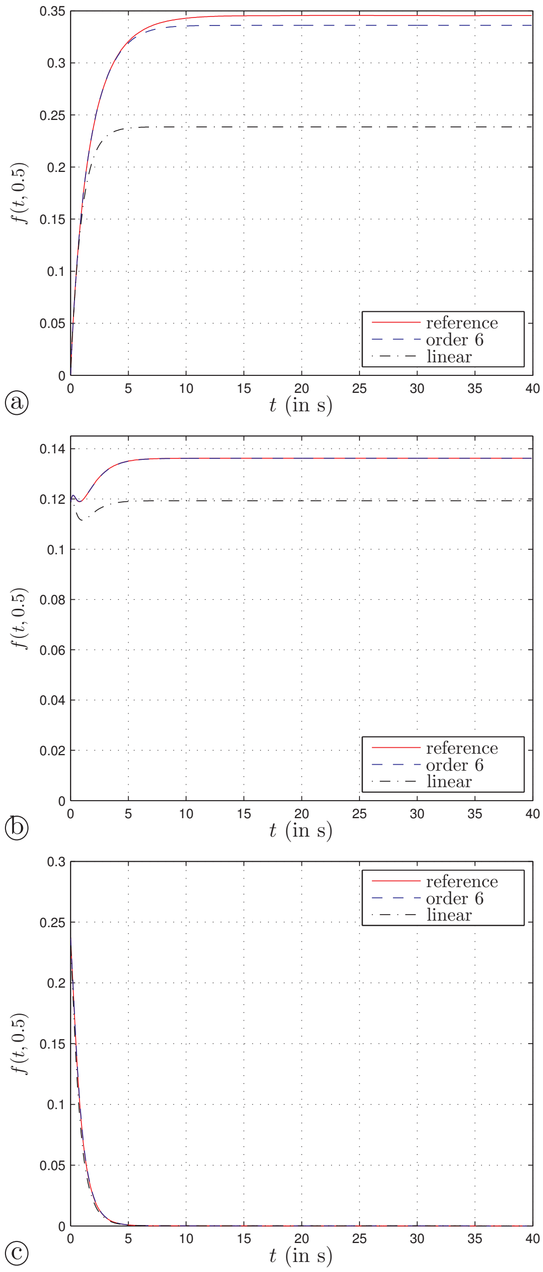

Figure 3. Simulations of $f(t, 0.5)$ for inputs and initial conditions such that $\|u\|_{\mathcal{U}} + \frac{1}{\omega}\|G(x_{\mathrm{ini}})\|_{\mathcal{X}} = 0.9\rho^{\star}$: ⓐ $x_{\mathrm{ini}} = 0$ (input only), ⓑ $\omega\|u\|_{\mathcal{U}}/\|G(x_{\mathrm{ini}})\|_{\mathcal{X}} = 1$ (equally balanced contributions of u and x_{ini}, ⓐ $u = 0$ (initial conditions only). These simulations are associated to marker$+$ in figure 1.

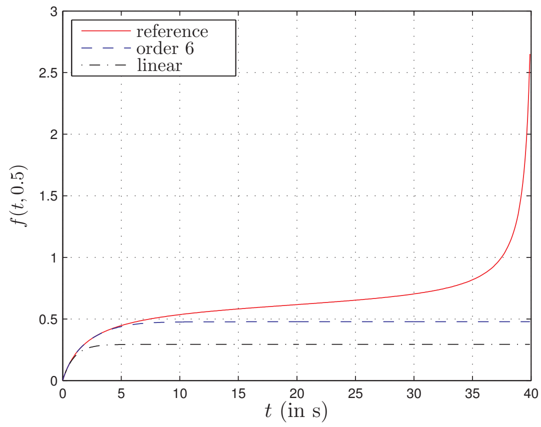

Figure 4. Simulations of $f(t, 0.5)$ for $x_{\mathrm{ini}} = 0$ and $u(t) = 1.11\rho^{\star}$ (to be compared to Fig. 3 ⓐ). These simulations are associated to marker$\times$ in figure 1.

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Appendix A. TECHNICAL LEMMA

Lemma 9. Let $A(X) = \sum_{k=1}^{+\infty} a_k X^k$ and $B(X) = \sum_{k=1}^{+\infty} b_k X^k$ be analytic functions at $X = 0$ with non-negative coefficients. Let $\beta \in \mathbb{R}_+^*$. Define $F(X) = \dfrac{\beta + B(X)}{1 - A(X)}$ and let $r \in \mathbb{R}_+^* \cup \{+\infty\}$ be the radius of convergence of F at $x = 0$. Then, the following results hold:

(i) At $x = 0$, F is nonzero and analytic with nonnegative Taylor coefficients.
(ii) Equation $x\,F'(x) - F(x) = 0$ has either one solution denoted σ (case 1) or zero solution (case 2), in $]0, r[$.
(iii) There exists a unique function $z \mapsto \Psi(z)$, analytic at $z = 0$ such that $\Psi(z) = z\,F\big(\Psi(z)\big)$. Its convergence radius ρ_Ψ at $z=0$ is such that

$$\text{(case 1)} \qquad \rho_\Psi \;=\; \rho^\star = \frac{\sigma}{F(\sigma)}, \tag{A.1}$$

$$\text{(case 2)} \qquad \rho_\Psi \;\geq\; \rho^\star = \lim_{x \to r^-} \frac{x}{F(x)}. \tag{A.2}$$

Proof.
Assertion (i): If $A = 0$, (i) is straightforward. Otherwise, A has at least one positive Taylor coefficients so that, for all $z \in \mathbb{C}$ such that $|z| < r$, $|A(z)| < A\big|z|\big) < \lim_{x \to r^-}(x) \leq 1$ and $F(z) = \big(\beta + B(z)\big) \sum_{n=0}^{+\infty} \big(A(z)\big)^n$, which proves (i).
Assertion (ii): Define $H(x) = x\,F'(x) - F(x)$ for $x \in [0, r[$. If F is affine then $H(x) = -\beta$ so that $x\,F'(x) - F(x) = 0$ has no solution. Otherwise, H is a strictly increasing function on $]0, r[$ from $H(0) < 0$ to $\ell = \lim_{x \to r^-} H(x) \in \mathbb{R} \cup \{+\infty\}$ since for all $x \in]0, r[$, $H'(x) = x\,F''(x) > 0$. Therefore, if $\ell > 0$, then $x\,F'(x) - F(x) = 0$ has a unique solution on $[0, r[$ (case 1), otherwise ($\ell \leq 0$), it has no solution (case 2).
Assertion (iii): In case 1, the hypotheses of the singular inversion theorem (see e.g. proposition IV.5. and theorem VI.6. in Flajolet and Sedgewick (2009)) are met, and its application proves (iii). In case 2, (iii) is a direct consequence of the analytic inversion lemma (see e.g. lemma 4.2. in Flajolet and Sedgewick (2009)).