Computation of convergence radius and error bounds of Volterra series for multiple input systems with an analytic nonlinearity in state

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***Abstract*— In this paper, the Volterra series decomposition of a class of multiple input time-invariant systems, analytic in state and affine in inputs is addressed. Computable bounds for the non-local-in-time convergence of the Volterra series to a trajectory of the system are given for infinite norms (Bounded Input Bounded Output results) and for specific weighted norms adapted to some "fading memory systems" (exponentially decreasing input-output results). This work extends results previously obtained for polynomial single input systems. Besides the increase in combinatorial complexity, a major difference with the single input case is that inputs may play different roles in the system behavior. Two types of inputs (called "principal" and "auxiliary") are distinguished in the convergence process to improve the accuracy of the bounds. The method is illustrated on the example of a frequency-modulated Duffing's oscillator.**

I. INTRODUCTION

Volterra series is a functional series expansion of the solution of nonlinear controlled systems introduced by the Italian mathematician Volterra [1]. This tool has been extensively used in signal processing, control, electronics, mechanics, acoustics, bio-medical engineering (etc) for modeling, identification and simulation purposes. There is a vast literature concerning Volterra series (see e.g. [2], [3], [4], [5], [6], [7], [8]) but only a few results on the convergence. These are mainly existence results (see [9] for complex linear analytic systems, [8], [10] for local-in-time results and [11] for fading memory systems). More recently, results in the frequency domain have been developed in [12], [13], results relying on regular perturbations (that can be related to Volterra series) are established in [14] and computable convergence radius for quadratic and polynomial systems are given in [15], [16].

This paper focuses on the computation of guaranteed convergence bounds for the input-to-state Volterra series expansion of a class of multiple-input systems, excited by bounded and exponentially decreasing signals. A major difference with the single input case studied in [16] is that two types of inputs, which play different roles in the convergence process, are introduced to characterize the convergence domain.

The paper is organized into six sections. Section II defines the notations, the functional setting and the class of systems under consideration. Section III recalls some results on Volterra series for single input systems. Section IV details the main results of the paper on convergence bounds for multiple input systems. In section V, these results are illustrated on an example. Section VI gives conclusions and perspectives.

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II. GENERAL FRAMEWORK

A. Notations and functional setting

The following notations are introduced, where $\mathbb{E}$, $\mathbb{E}_1$, ..., $\mathbb{E}_K$ $(K\geq 2)$ and $\mathbb{F}$ are real normed vector spaces:

- $\mathcal{L}(\mathbb{E},\mathbb{F})$ is the vector space of continuous linear functions from a $\mathbb{E}$ to $\mathbb{F}$ with norm $\|f\|_{\mathcal{L}(\mathbb{E},\mathbb{F})} = \sup_{x\in B_{\mathbb{E}}}\|f(x)\|_{\mathbb{F}}$ where $B_{\mathbb{E}}$ is the unit ball in $\mathbb{E}$.
- $\mathcal{ML}(\mathbb{E}_1,\dots,\mathbb{E}_K,\mathbb{F})$ is the vector space of continuous multilinear functions $f:\mathbb{E}_1\times\ldots\times\mathbb{E}_K\to\mathbb{F}$ where
$$\|f\|_{\mathcal{ML}(\mathbb{E}_1,\dots,\mathbb{E}_K,\mathbb{F})} = \sup_{(x_1,\dots,x_K)\in B_{\mathbb{E}_1}\times\ldots\times B_{\mathbb{E}_K}}\|f(x_1,\dots,x_K)\|_{\mathbb{F}}.$$
- $\mathcal{ML}_{j_1,\dots,j_K}(\mathbb{E}_1,\dots,\mathbb{E}_K,\mathbb{F})$ is the concise notation for $\mathcal{ML}(\underbrace{\mathbb{E}_1,\dots,\mathbb{E}_1}_{j_1},\underbrace{\mathbb{E}_2,\dots,\mathbb{E}_2}_{j_2},\dots,\underbrace{\mathbb{E}_K,\dots,\mathbb{E}_K}_{j_K},\mathbb{F})$.

Moreover, $\mathbb{X}=\mathbb{R}^n$ $(n\in\mathbb{N}^*)$ is a real vector space equipped with a norm $\|.\|_{\mathbb{X}}$ and $\mathbb{T}$ denotes the time interval $[0,T]$ with $T>0$ or $\mathbb{R}_+$. For all $\lambda\in\mathbb{R}_+$ and $m\in\mathbb{N}^*$, we introduce:

- $\mathcal{X}_\lambda$ is the set of functions f such that $t\mapsto \mathrm{e}^{\lambda t}f(t)\in\mathcal{L}^\infty\big(\mathbb{T},\mathbb{X}\big)$ endowed with the norm
$$\|f\|_{\mathcal{X}_\lambda} = \sup_{t\in\mathbb{T}}\Big(\mathrm{e}^{\lambda t}\,\|f(t)\|_{\mathbb{X}}\Big).$$
- $\mathcal{U}_\lambda$ is defined like $\mathcal{X}_\lambda$, replacing $\mathbb{X}$ by $\mathbb{R}$.
- $\mathcal{V}^m_\lambda$ is the set of functions $f:\mathbb{T}\times\mathbb{T}^m\to\mathbb{X}$ such that $t\mapsto\big(\tau\mapsto \mathrm{e}^{\lambda t-\lambda\overline{\tau}}f(t,\tau)\big)\in\mathcal{L}^\infty\big(\mathbb{T},\mathcal{L}^1\big(\mathbb{T}^m,\mathbb{X}\big)\big)$, where $\forall\tau=(\tau_1,\dots,\tau_m)\in\mathbb{T}^m$, $\overline{\tau}=\tau_1+\tau_2+\cdots+\tau_m$. This set is endowed with the norm defined
$$\|f\|_{\mathcal{V}^m_\lambda} = \sup_{t\in\mathbb{T}}\Big(\mathrm{e}^{\lambda t}\int_{\mathbb{T}^m}\|f(t,\tau)\|_{\mathbb{X}}\,\mathrm{e}^{-\lambda\overline{\tau}}\mathrm{d}\tau\Big).$$
- $\mathcal{VS}_\lambda$ is the set of the series $(f_m)_{m\in\mathbb{N}^*}$ such that for all $m\in\mathbb{N}^*$, $f_m\in\mathcal{V}^m_\lambda$.

Remark 1: $\mathcal{U}_\lambda$ is the set of bounded signals decreasing at least like $\mathrm{e}^{-\lambda t}$. If $\lambda_2>\lambda_1\geq 0$ then $\mathcal{U}_{\lambda_2}\subset\mathcal{U}_{\lambda_1}\subseteq\mathcal{U}_0$.

B. Systems under consideration

The systems under consideration are analytic in state $x:\mathbb{T}\to\mathbb{X}$, affine in input $u:\mathbb{T}\to\mathbb{R}^d$ and described on $\mathbb{T}$ by

$$\dot{x} \;=\; f(x,u) = A\,x + B\,u + P(x) + Q(x,u), \quad (1)$$

with zero initial conditions $x(0)=0$, where A is a $n\times n$ real matrix, B is a nonzero $n\times d$ real matrix, and where P and Q are expressed as a series of homogeneous contributions

$$P(x)=\sum_{k=2}^{\infty}P_k(\underbrace{x,\dots,x}_{k}),\quad Q(x,u)=\sum_{k=2}^{\infty}Q_k(\underbrace{x,\dots,x}_{k-1},u), \quad (2)$$

with $P_k\in\mathcal{ML}_k(\mathbb{X},\mathbb{X})$ and $Q_k\in\mathcal{ML}_{k-1,1}\big(\mathbb{X},\mathbb{R}^d,\mathbb{X}\big)$.

Remark 2: If (x, u) is in the analytic domain of a function g almost everywhere (a.e.), the output $y = g(x, u)$ is bounded a.e. Hence, we focus on the study of input-to-state relations.

III. RECALLS ON SINGLE INPUT SYSTEMS

This section recalls some results established in [16].

A. Definitions and formal solution

Definition 1 (Volterra series in $\mathcal{VS}_\lambda$): A causal SI-system can be described by an input-to-state Volterra series $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}_\lambda$ if there exist $\rho \in \mathbb{R}_+^*$ such that for all input $u \in \mathcal{U}_\lambda$ satisfying $\|u\|_{\mathcal{U}_\lambda} < \rho$ and $t \in \mathbb{T}$, the series

$$\forall t \in \mathbb{T}, \quad x(t) = \sum_{m\in\mathbb{N}^*} \int_{[0,t]^m} h_m(t,\tau) \left[\Pi_m u\right](\tau) \, \mathrm{d}\tau, \quad (3)$$

with $\left[\Pi_m u\right](\tau) = \prod_{i=1}^m u(\tau_i)$, is normally convergent in $\mathcal{X}_\lambda$. The function h_m is called the kernel of order m.

Definition 2 (Index set and selection function): Let $m \in \mathbb{N}^*$ and $K \in \mathbb{N}^*$. The set $\mathbb{M}_m^K$ is defined by

$$\mathbb{M}_m^K = \left\{ p \in (\mathbb{N}^*)^K \mid p_1 + \cdots + p_K = m \right\}.$$

For all $p \in \mathbb{M}_m^K$ and for all $k \in [1, K]_\mathbb{N}$, the selection function $S_p^k : \mathbb{T}^m \to \mathbb{T}^{p_k}$ is defined by, denoting $\tau = (\tau_1, \tau_2, \dots, \tau_m)$,

$$S_p^k(\tau) = \left(\tau_{p_1+\cdots+p_{k-1}+1}, \tau_{p_1+\cdots+p_{k-1}+2}, \dots, \tau_{p_1+\cdots+p_k}\right).$$

Note that if $K > m$, then $\mathbb{M}_m^K = \emptyset$.

Proposition 1 (Kernels recursive construction): Let the family of kernels $\{h_m\}_{m\in\mathbb{N}^*}$ be defined by $h_1 : \mathbb{T}\times\mathbb{T} \to \mathbb{X}$ $h_1(t,\tau_1) = 1_{\mathbb{R}_+}(t-\tau_1)\,\mathrm{e}^{A(t-\tau_1)} B$, and by $h_m : \mathbb{T}\times\mathbb{T}^m \to \mathbb{X}$,

$$h_m(t,\tau) = 1_{\mathbb{R}_+}(t - \max\tau) \left(\int_{\max\tau}^t v_m(t,\theta,\tau)\,\mathrm{d}\theta + w_m(t,\tau) \right)$$

if $m \geq 2$, where $1_{\mathbb{R}_+}$ denotes the Heaviside function and

$$v_m(t,\theta,\tau) = \mathrm{e}^{A(t-\theta)} \sum_{k=2}^m \sum_{p\in\mathbb{M}_m^k} P_k\Big(h_{p_1}\big(\theta, S_p^1(\tau)\big), \dots$$
$$\dots, h_{p_k}\big(\theta, S_p^k(\tau)\big)\Big), \quad (4)$$

$$w_m(t,\tau) = 1_{\mathbb{R}_+}(\tau_m - \max_{1\leq i<m} \tau_i)\, \mathrm{e}^{A\,(t-\tau_m)} \left[\sum_{k=2}^m \sum_{\substack{q\in\mathbb{M}_m^k \\ q_k = 1}} \right.$$
$$\left. Q_k\Big(h_{q_1}\big(\tau_m, S_q^1(\tau)\big), \dots, h_{q_{k-1}}\big(\tau_m, S_q^{k-1}(\tau)\big), 1\Big) \right]. \quad (5)$$

Then, the Volterra series (3) is a formal solution of system (1-2).

B. Gain bound function and theoretical convergence result

Definition 3 (Gain bound function φ_λ): Let $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}_\lambda$ and $\rho \in \mathbb{R}_+$ be the convergence radius of the formal series $\sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}_\lambda^m} X^m$. If $\rho > 0$, then the *gain bound function* φ_λ of $\{h_m\}_{m\in\mathbb{N}^*}$ is defined for all $z \in \mathbb{C}$ such that $|z| < \rho$ by

$$\varphi_\lambda(z) = \sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}_\lambda^m} z^m.$$

Theorem 1: Let $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}_\lambda$ be such that φ_λ has a non zero convergence radius $\rho > 0$. Then, the Volterra series is convergent in $\mathcal{X}_\lambda$ for inputs such that $\|u\|_{\mathcal{U}_\lambda} < \rho_\lambda$. In this case, $x \in \mathcal{X}_\lambda$ satisfies $\|x\|_{\mathcal{X}_\lambda} \leq \varphi\big(\|u\|_{\mathcal{U}_\lambda}\big) < \infty$.

C. Computable results and guaranteed error bounds

In [16], the following results (with examples) were presented in the case where P, Q are polynomials, $\mathbb{T} = \mathbb{R}_+$, A is Hurwitz with $-a = \max\big(\Re\mathrm{e}(\operatorname{spec} A)\big) < 0$, and $\lambda \in [0, a)$.

Proposition 2 (Coefficients $\kappa_{k,\lambda}$ and norm of h_1): Let $\beta > 0$ be such that for all $t \in \mathbb{T}$, $\left\|\mathrm{e}^{At}\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \leq \beta\,\mathrm{e}^{-at}$. Then, the coefficients defined by, for all $k \in \mathbb{N}^*$,

$$\kappa_{k,\lambda} = \sup_{t\in\mathbb{T}} \left(\mathrm{e}^{\lambda t} \int_0^t \left\|\mathrm{e}^{A(t-\theta)}\right\|_{\mathcal{L}(\mathbb{X},\mathbb{X})} \mathrm{e}^{-k\lambda\theta}\,\mathrm{d}\theta \right) \quad (6)$$

are finite and satisfy $0 < \kappa_{k,\lambda} \leq \kappa_{1,\lambda} \leq \frac{\beta}{a-\lambda}$. Moreover, $h_1 \in \mathcal{V}_\lambda^1$ and $\|h_1\|_{\mathcal{V}_\lambda^1} \leq \frac{\beta}{a-\lambda} \|B\|_{\mathcal{L}(\mathbb{R},\mathbb{X})}$.

Definition 4: The function $\mathcal{F}_\lambda$ is formally defined by

$$\mathcal{F}_\lambda(X) = \frac{\|h_1\|_{\mathcal{V}_\lambda^1} + \displaystyle\sum_{k=2}^{\deg(Q)} \mathcal{Q}_k\, X^{k-1}}{1 - \displaystyle\sum_{k=2}^{\deg(P)} \mathcal{P}_k\, X^{k-1}},$$

with $\mathcal{P}_k = \kappa_{k,\lambda}\big\|P_k\big\|_{\mathcal{ML}_k(\mathbb{X},\mathbb{X})}$ and $\mathcal{Q}_k = \kappa_{k,\lambda}\big\|Q_k\big\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{R},\mathbb{X})}$.

Theorem 2 (Lower bound for the convergence radius): The family $\{h_m\}_{m\in\mathbb{N}^*}$ defined in proposition 1 belongs to $\mathcal{VS}_\lambda$. Moreover, the convergence radius of its gain bound function is greater than $\rho_\lambda^\star$, where $\rho_\lambda^\star > 0$ is given by

$$\rho_\lambda^\star = \lim_{x\to+\infty} \frac{x}{\mathcal{F}_\lambda(x)}, \qquad \text{if } \mathcal{F}_\lambda \text{ in (4) is affine} \quad (7)$$

$$\rho_\lambda^\star = \frac{\sigma_\lambda}{\mathcal{F}_\lambda(\sigma_\lambda)}, \qquad \text{otherwise.} \quad (8)$$

In (8), σ_λ is the unique solution of $\mathcal{F}_\lambda(\sigma) - \sigma\,\mathcal{F}'_\lambda(\sigma) = 0$ on $]0, R[$ where R is the convergence radius of $\mathcal{F}_\lambda$ at $x = 0$.

Theorem 3 (Truncation error bound): Assume that $\mathcal{F}_\lambda$ is not affine. Let σ_λ and $\rho_\lambda^\star$ be defined as in theorem 2. For all $M \in \mathbb{N}^*$, let $V_M x(t) = \sum_{m=1}^M \int_{[0,t]^m} h_m(t,\tau) \left[\Pi_m u\right](\tau)\,\mathrm{d}\tau$ denote the finite M-order partial sum of the Volterra series. Then, for all $u \in \mathcal{D}_\lambda^\star$, $\big\|x - V_M x\big\|_{\mathcal{X}_\lambda} \leq \sigma_\lambda \dfrac{\big(\|u\|_{\mathcal{U}_\lambda}/\rho_\lambda^\star\big)^{M+1}}{1 - \|u\|_{\mathcal{U}_\lambda}/\rho_\lambda^\star}$.

IV. MULTIPLE INPUT ANALYTIC SYSTEMS

In this section, results of section III are extended to multiple-input (MI) analytic-in-state systems. Most of the definitions and key points can be straightforwardly generalized (see IV-A and IV-B). However, when some columns of matrix B are zero, the corresponding inputs do not influence the system as long as other inputs are zero (section IV-C). This defines two types of inputs (called *auxiliary* and *principal*) which play different roles in the convergence process and in the results and proofs given below (see IV-D).

In the sequel, the number of inputs is $d \geq 2$.

A. Definitions and formal solution

Definition 5: Let $\mathcal{VS}_\lambda^d$ be the set of the series $\{f_m\}_{m\in\mathbb{N}_d^*}$ indexed by the multiple orders m belonging to $\mathbb{N}_d^* = \mathbb{N}^d \backslash \{0\}$ and such that, for all $m \in \mathbb{N}_d^*$,

$$f_m \in \mathcal{V}_\lambda^{\overline{m}}, \text{ denoting } \overline{m} = m_1 + \cdots + m_d.$$

A causal MI-system can be described by an input-to-state Volterra series $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}_\lambda^d$ if there exist a non empty domain $\mathcal{D} \subseteq (\mathcal{U}_\lambda)^d$ such that for all input $u \in \mathcal{D}$, the series

$$\forall t \in \mathbb{T}, \quad x(t) = \sum_{m\in\mathbb{N}_d^*} \int_{[0,t]^{\overline{m}}} h_m(t,\tau) \left[\Pi_m u\right](\tau)\,\mathrm{d}\tau, \quad (9)$$

with $\left[\Pi_m u\right](\tau) = \prod_{i=1}^{d}\prod_{j=1}^{m_i} u_i(\tau_{m_1+\cdots+m_{i-1}+j})$, is normally convergent in $\mathcal{X}_\lambda$.

Definition 6 (Multiple-index set and selection function): Let $m \in \mathbb{N}_d^*$ and $K \in \mathbb{N}^*$. The set $\mathbb{M}_m^K$ is defined by

$$\mathbb{M}_m^K = \left\{p \in \mathbb{N}^{d\times K} \,\middle|\, \text{all columns of } p \text{ belong to } \mathbb{N}_d^* \text{ and their sum equals to } m\right\}.$$

Moreover, for all $p \in \mathbb{M}_m^K$ and for all $k \in [1,K]_\mathbb{N}$, the selection function is defined by

$$S_p^k : \tau \in \mathbb{T}^{\overline{m}} \to (\tau_{\ell_1},\ldots,\tau_{\ell_L}) \in \mathbb{T}^L$$

where $L = \sum_{i=1}^d p_{i,k}$ and the sequence of indexes $\ell_1,\ldots,\ell_L$ is given by

$$\begin{array}{llll} \Lambda_{1,k}{+}1\,, & \Lambda_{1,k}{+}2\,, & \ldots\ldots\,, & \Lambda_{1,k}{+}p_{1,k}\,, \\ \Lambda_{2,k}{+}1\,, & \Lambda_{2,k}{+}2\,, & \ldots\ldots\,, & \Lambda_{2,k}{+}p_{2,k}\,, \\ & & \vdots & \\ \Lambda_{d,k}{+}1\,, & \Lambda_{d,k}{+}2\,, & \ldots\ldots\,, & \Lambda_{d,k}{+}p_{d,k}\,, \end{array}$$

denoting $\Lambda_{i,k} = \sum_{i'=1}^{i-1}\sum_{k'=1}^{K} p_{i',k'} + \sum_{k'=1}^{k-1} p_{i,k'}$.

Note that, in this sequence, the i-th row is empty if $p_{i,k} = 0$. But, the complete sequence is not empty since $L \geq 1$ from the definition of $\mathbb{M}_m^K$. Note also that if $K > \overline{m}$, then $\mathbb{M}_m^K = \emptyset$.

Proposition 3 (Kernels recursive construction): For all $n \in [1,d]_\mathbb{N}$, let $e_n \in \mathbb{R}^d$ be the vector composed of zeros except the n-th coordinate which equals to 1, so that $\{e_1,\ldots,e_d\}$ is the canonical basis of $\mathbb{R}^d$. Moreover, let the family of kernels $\{h_m\}_{m\in\mathbb{N}_d^*}$ be defined by $h_{e_n} : \mathbb{T}\times\mathbb{T} \to \mathbb{X}$, $h_{e_n}(t,\tau_1) = 1_{\mathbb{R}_+}(t-\tau_1)\,\mathrm{e}^{A(t-\tau_1)} B\, e_n$, if $m = e_n$ with $n \in [1,d]_\mathbb{N}$, and by $h_m : \mathbb{T}\times\mathbb{T}^{\overline{m}} \to \mathbb{X}$, $h_m(t,\tau) = 1_{\mathbb{R}_+}(t - \max\tau)\left(\int_{\max\tau}^t v_m(t,\theta,\tau)\,\mathrm{d}\theta + w_m(t,\tau)\right)$, if $\overline{m} \geq 2$, where, denoting $p_{*,k}$ the k-th column of p,

$$v_m(t,\theta,\tau) = \mathrm{e}^{A(t-\theta)} \sum_{k=2}^{\overline{m}} \sum_{p\in\mathbb{M}_m^k} P_k\Big(h_{p_{*,1}}\big(\theta, S_p^1(\tau)\big),\ldots \ldots, h_{p_{*,k}}\big(\theta, S_p^k(\tau)\big)\Big), \quad (10)$$

$$w_m(t,\tau) = 1_{\mathbb{R}_+}(\tau_{\overline{m}} - \max_{1\leq j<\overline{m}} \tau_j) \left[\sum_{k=2}^{\overline{m}} \sum_{\left\{\substack{q\in\mathbb{M}_m^k \\ q_{*,k}=1}\right.} \mathrm{e}^{A\left(t - S_q^k(\tau)\right)} Q_k\Big(h_{q_{*,1}}\big(S_q^k(\tau), S_q^1(\tau)\big),\ldots, h_{q_{*,k-1}}\big(S_q^k(\tau), S_q^{k-1}(\tau)\big), q_{*,k}\Big)\right]. \quad (11)$$

Then, the Volterra series (9) is a formal solution of system (1-2).

B. Gain bound function and theoretical convergence results

Definition 7 (Multi-variate gain bound function φ_λ): Let $\{h_m\}_{m\in\mathbb{N}^*} \in \mathcal{VS}_\lambda^d$. Suppose that there exists a neighborhood $V \subseteq \mathbb{C}^d$ of 0 such that the formal multi-variate series $(X_1,\ldots,X_d) \mapsto \sum_{m\in\mathbb{N}^*} \|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}} X_1^{m_1}\ldots X_d^{m_d}$ is normally convergent. Then, the *gain bound function* φ_λ is defined by

$$\varphi_\lambda(z) = \sum_{m\in\mathbb{N}_d^*} \|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}} \prod_{i=1}^d z^{m_i}.$$

Theorem 4: For all input u belonging to

$$\Upsilon(V) = \left\{u \in \mathcal{U}_\lambda^d \,\middle|\, \left(\|u_1\|_{\mathcal{U}_\lambda},\ldots,\|u_d\|_{\mathcal{U}_\lambda}\right) \in V \cap \mathbb{R}_+^d\right\}, \quad (12)$$

the Volterra series is convergent in $\mathcal{X}_\lambda$ and

$$\|x\|_{\mathcal{X}_\lambda} \leq \varphi_\lambda\big(\|u_1\|_{\mathcal{U}_\lambda},\ldots,\|u_d\|_{\mathcal{U}_\lambda}\big) < \infty.$$

Proof: Let $u \in \Upsilon(V)$. Then, $z = \big(\|u_1\|_{\mathcal{U}_\lambda},\ldots,\|u_d\|_{\mathcal{U}_\lambda}\big)$ belongs to V and $\varphi_\lambda\big(\|u_1\|_{\mathcal{U}_\lambda},\ldots,\|u_d\|_{\mathcal{U}_\lambda}\big) < \infty$. Now, for all $m \in \mathbb{N}_d^*$,

$$\begin{aligned} &\sup_{t\in\mathbb{T}} \left(\mathrm{e}^{\lambda t} \Big\| \int_{[0,t]^{\overline{m}}} h_m(t,\tau)\left[\Pi_m u\right](\tau)\,\mathrm{d}\tau \Big\|_{\mathbb{X}}\right) \\ &\quad\leq \sup_{t\in\mathbb{T}} \left(\mathrm{e}^{\lambda t} \int_{[0,t]^{\overline{m}}} \big\|h_m(t,\tau)\big\|_{\mathbb{X}}\, \mathrm{e}^{-\lambda\overline{\tau}} \prod_{i=1}^d \big(\|u_i\|_{\mathcal{U}_\lambda}\big)^{m_i}\,\mathrm{d}\tau\right) \\ &\quad\leq \|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}} \prod_{i=1}^d \big(\|u_i\|_{\mathcal{U}_\lambda}\big)^{m_i}. \end{aligned}$$

Hence, the series $\sum_{m\in\mathbb{N}_d^*} \int_{[0,t]^{\overline{m}}} h_m(t,\tau)\left[\Pi_m u\right](\tau)\,\mathrm{d}\tau$ converges normally in $\mathcal{X}_\lambda$ to a limit x such that

$$\|x\|_{\mathcal{X}_\lambda} = \sup_{t\in\mathbb{T}} \left(\Big\| \sum_{m\in\mathbb{N}_d^*} \int_{[0,t]^{\overline{m}}} h_m(t,\tau)\left[\Pi_m u\right](\tau)\,\mathrm{d}\tau \Big\|_{\mathbb{X}}\right) \leq \sum_{m\in\mathbb{N}_d^*} \|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}} \prod_{i=1}^d \big(\|u_i\|_{\mathcal{U}_\lambda}\big)^{m_i} \leq \varphi_\lambda\big(\|u_1\|_{\mathcal{U}_\lambda},\ldots,\|u_d\|_{\mathcal{U}_\lambda}\big). \qquad \blacksquare$$

C. Principal and auxiliary inputs

Hypothesis 1: The system (1-2) is such that $t \mapsto \exp(At)$ belongs $\mathcal{L}^1(\mathbb{T},\mathbb{R}^{n\times n})$, that is: A is supposed to be Hurwitz if $\mathbb{T} = \mathbb{R}_+$ and there is no assumption on A if $\mathbb{T}$ is a finite interval $[0,T]$. Then, we assume that $\lambda \in \mathbb{R}_+$. Moreover, if $\mathbb{T} = \mathbb{R}_+$, then $\lambda < a$ where $-a = \max\big(\Re\mathrm{e}(\operatorname{spec} A)\big) < 0$.

Denote $\{e_1,\ldots,e_d\}$ the canonical basis of $\mathbb{R}^d$. For all $n \in [0,d]_\mathbb{N}$, the kernel h_{e_n} given in proposition 3 belong to $\mathcal{V}_\lambda^1$ and is zero if $B\,e_n = 0$. We assume that the inputs are sorted so that non-zero columns of B come first and are indexed from 1 to d_π ($d_\pi \geq 1$ since $B \neq 0$) and the remaining d_α null columns (if any) are indexed from $d_\pi + 1$ to d. Hence, whenever the first d_π inputs are set to zero, the system will remain at the null steady-state, even if the d_α last inputs are nonzero. These two types of inputs play different roles in the convergence of the Volterra series. They are called *principal* for the first ones (index π), and *auxiliary* for the second ones (index α).

The effect on the convergence will be accounted for through a ratio γ between the norms of auxiliary and principal inputs. This ratio appears in (14) below.

Definition 8 (Function $\mathcal{F}_{\lambda,\gamma}$): Define the $d_\pi \times d$ matrix $\chi_\pi = \big(I_{d_\pi}\,,\,0_{d_\pi,d_\alpha}\big)$, the $d_\alpha \times d$ matrix $\chi_\alpha = \big(0_{d_\alpha,d_\pi}\,,\,I_{d_\alpha}\big)$ and the $d\times d$ matrices $W_\pi = {\chi_\pi}^T\,(\mathrm{diag}\,\omega)^{-1}\chi_\pi$ and $W_\alpha = {\chi_\alpha}^T\,\chi_\alpha$ where the vector $\omega = \big(\omega_1,\ldots,\omega_{d_\pi}\big)^T$ is chosen such that

$$\forall n \in [1,d_\pi]_{\mathbb{N}}, \quad \omega_n \geq \|h_{e_n}\|_{\mathcal{V}^1_\lambda}. \tag{13}$$

For all $\gamma \in \mathbb{R}_+$, the function $\mathcal{F}_{\lambda,\gamma}$ is formally defined by

$$\mathcal{F}_{\lambda,\gamma}(X) = \frac{1 + \sum_{k=2}^{+\infty} \mathfrak{p}_k\, X^{k-1} + \gamma \sum_{k=2}^{+\infty} \mathfrak{a}_k\, X^{k-1}}{1 - \sum_{k=2}^{+\infty} \mathcal{P}_k\, X^{k-1}}, \tag{14}$$

where $\mathcal{P}_k = \kappa_{k,\lambda}\,\|P_k\|_{\mathcal{ML}_k(\mathbb{X},\mathbb{X})}$ and, for $\zeta = \pi$ or α, defining $\widetilde{Q}^\zeta_k(x_1,\ldots,x_{k-1},u) = Q_k\big(x_1,\ldots,x_{k-1},W_\zeta\,u\big)$,

$$\begin{aligned}\mathfrak{p}_k &= \kappa_{k,\lambda}\,\big\|\widetilde{Q}^\pi_k\big\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{R}^d,\mathbb{X})}\,,\\ \mathfrak{a}_k &= \kappa_{k,\lambda}\,\big\|\widetilde{Q}^\alpha_k\big\|_{\mathcal{ML}_{k-1,1}(\mathbb{X},\mathbb{R}^d,\mathbb{X})}\,.\end{aligned}$$

Remark 3: If there is no auxiliary input ($d_\alpha = 0$), then $\mathfrak{a}_k = 0$ and $\mathcal{F}_{\lambda,\gamma}$ does not depend on γ.

D. Computable results and guaranteed error bounds

Theorem 5 (Convergence subset): Let $\lambda \in \mathbb{R}_+$ be such that hypothesis 1 is satisfied. Let $\gamma \in \mathbb{R}_+$. Then, the family $\{h_m\}_{m\in\mathbb{N}^\star_d}$ defined in proposition 3 belongs to $\mathcal{VS}^d_\lambda$. Let $R \in \mathbb{R}^*_+ \cup \{+\infty\}$ be the convergence radius of $\mathcal{F}_{\lambda,\gamma}$ at $x = 0$. Equation $x\,\mathcal{F}'_{\lambda,\gamma}(x) - \mathcal{F}_{\lambda,\gamma}(x) = 0$ has either one solution denoted $\sigma_{\lambda,\gamma}$ (case 1) or zero solution (case 2), in $]0,R[$. Let $\rho^\star_{\lambda,\gamma} > 0$ be defined by

$$\text{(case 1)} \qquad \rho^\star_{\lambda,\gamma} = \frac{\sigma_{\lambda,\gamma}}{\mathcal{F}_{\lambda,\gamma}(\sigma_{\lambda,\gamma})}, \tag{15}$$

$$\text{(case 2)} \qquad \rho^\star_{\lambda,\gamma} = \lim_{x\to R^-} \frac{x}{\mathcal{F}_{\lambda,\gamma}(x)}. \tag{16}$$

Then, the Volterra series is convergent in $\mathcal{X}_\lambda$ for all input u belonging to $\Upsilon(V_\lambda)$ where $V_\lambda = \bigcup_{\gamma\in\mathbb{R}} V_{\lambda,\gamma}$ and

$$V_{\lambda,\gamma} = \Big\{ z \in \mathbb{C}^d \,\Big|\, \sum_{n=1}^{d_\pi} \omega_n\,|z_n| < \rho^\star_{\lambda,\gamma} \text{ and } \sum_{n=d_\pi+1}^{d} z_n = \gamma \sum_{n=1}^{d_\pi} \omega_n\, z_n \Big\}, \tag{17}$$

and ω is given in (13).
The proof is detailed in appendix B.

Theorem 6 (Truncation error bound): Let $M \in \mathbb{N}^*$ and consider the finite M-order partial sum of the Volterra series

$$S_M x(t) = \sum_{\begin{cases} m \in \mathbb{N}^\star_d \\ \overline{m} \leq M\end{cases}} \int_{[0,t]^{\overline{m}}} h_m(t,\tau)\,\big[\Pi_m u\big](\tau)\,\mathrm{d}\tau.$$

Let $\gamma \in \mathbb{R}^*_+$, $u \in \Upsilon(V_\gamma)$, $\beta = \rho^\star_{\lambda,\gamma}\Big(\frac{1}{d_\pi\omega_1},\ldots,\frac{1}{d_\pi\omega_{d_\pi}},\overbrace{\frac{\gamma}{d_\alpha},\ldots,\frac{\gamma}{d_\alpha}}^{d_\alpha}\Big)^T$.
In case 1 ($\exists!\,\sigma_{\lambda,\gamma} < +\infty$), if $\sup_{1\leq n\leq d} \frac{\|u_n\|_{\mathcal{U}_\lambda}}{\beta_n} = U < 1$, then

$$\big\|x - S_M x\big\|_{\mathcal{X}_\lambda} \leq \sigma_{\lambda,\gamma}\, U^{M+1}/(1-U).$$

The proof is detailed in appendix C.

V. AN EXAMPLE

To illustrate these results, an example is presented in the BIBO case ($\lambda = 0$). Let $a \in]0,1[$, $\varepsilon \in \mathbb{R}^*_+$, and consider the system governed by

$$\forall t > 0, \qquad \ddot{y} + 2a\dot{y} + (1 + \varepsilon y^2 + u_2)\,y = u_1, \tag{18}$$

with zero initial conditions $y(0) = 0$, $\dot{y}(0) = 0$ and where the input is $u = [u_1,u_2]^T$. This system defines a damped Duffing oscillator where u_1 is an exciting signal and u_2 can be interpreted as a frequency-modulation signal. It takes the form (1-2) where $\mathbb{X} = \mathbb{R}^2$ is associated with the euclidean norm, the state is $x = [y,\dot{y}]^T$, $d = 2$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2a \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$P(x) = \begin{pmatrix} 0 \\ -\varepsilon x_1^3 \end{pmatrix}, \qquad Q(x,u) = \begin{pmatrix} 0 \\ -x_1 u_2 \end{pmatrix},$$

so that $\max\Big(\Re\mathrm{e}\big(\mathrm{spec}(A)\big)\Big) = -a < 0$. Following section IV-C, input u_1 is principal and u_2 is auxiliary.

In definition 8, straightforward computations yield for all $k \in \mathbb{N}^*$, $\mathfrak{p}_k = 0$, $\mathfrak{a}_2 = \kappa_{2,0}$, $\mathcal{P}_2 = \varepsilon\kappa_{2,0}$ and $\mathfrak{a}_k = \mathcal{P}_k = 0$ if $k \neq 2$. In this example, $P(x)$ and B_1 are collinear ($P(x) = -\varepsilon x_1^3 B_1$) so $\kappa_{2,0}$ can be replaced by $\nu = \|h_{e_1}\|_{\mathcal{V}^1}$ (which is smaller) in the proof of theorem 5. This leads to

$$\mathcal{F}_{0,\gamma}(X) = \frac{1+\gamma\nu X}{1-\varepsilon\nu X^2}.$$

Then, $\sigma_{0,\gamma}$ is the positive root of $2\varepsilon\gamma\nu^2\,X^3 + 3\varepsilon\nu\,X^2 - 1$ and

$$\rho^\star_{0,\gamma} = \sigma_{0,\gamma}\,\frac{1-\varepsilon\nu\,(\sigma_{0,\gamma})^2}{1+\gamma\nu\,\sigma_{0,\gamma}}. \tag{19}$$

For $\gamma = 0$, (19) yields $\rho^\star_{0,\gamma} = \frac{2}{3\sqrt{3\nu\varepsilon}}$ and, for $\gamma \to +\infty$, $\sigma_{0,\gamma} \sim (2\varepsilon\nu^2\gamma)^{-1/3}$ and $\rho^\star_{0,\gamma} \sim (\nu\gamma)^{-1}$. From theorem 5, for a given $\gamma \geq 0$, the Volterra series converges if $(\|u_1\|,\|u_2\|)$ belongs to the line segment

$$\|u_1\|_{\mathcal{U}_\lambda} < \frac{\rho^\star_{0,\gamma}}{\omega} \quad\text{and}\quad \|u_2\|_{\mathcal{U}_\lambda} = \gamma\omega\,\|u_1\|_{\mathcal{U}_\lambda},$$

for $\omega \geq \|h_{e_1}\|_{\mathcal{V}^1_0}$. Hence, the union of all these segments is the region under the curve described by $(x,y) = (\rho^\star_{0,\gamma}/\omega, \gamma\rho^\star_{0,\gamma})$ for $\gamma \geq 0$. Figure 1 shows such curves for the system with parameters $a = 0.65$ and $\varepsilon = 0.1$.

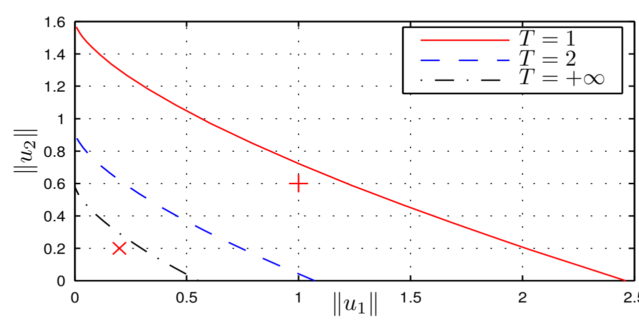

Fig. 1. Convergence domain boundaries for $\mathbb{T} = [0,T]$ with $T \in \{1,2,+\infty\}$. For $T = 1$, $\nu = \|h_{e_1}\|_{\mathcal{V}^1_0} \approx 0.63$. For $T = 2$, $\nu \approx 1.09$. For $T = +\infty$, $\nu \approx 1.69$. In each case, we have chosen $\omega = \nu$ and the curve $(x,y) = (\rho^\star_{0,\gamma}/\omega, \gamma\rho^\star_{0,\gamma})$ is parameterized by $\gamma \in \mathbb{R}_+$.

Time simulations based on the first seven Volterra kernels are shown in figure 2. Curve 1 corresponds to × in figure 1, for which the Volterra series expansion is convergent for any time horizon T. In practice, an accurate approximation is obtained at order 3. Curve 2 corresponds to + in figure 1. From figure 1, the convergence is guaranteed for $T=1$ but neither for $T=2$ nor $T=+\infty$. In figure 2, the divergence seems to appear before order 7 for $T>6$.

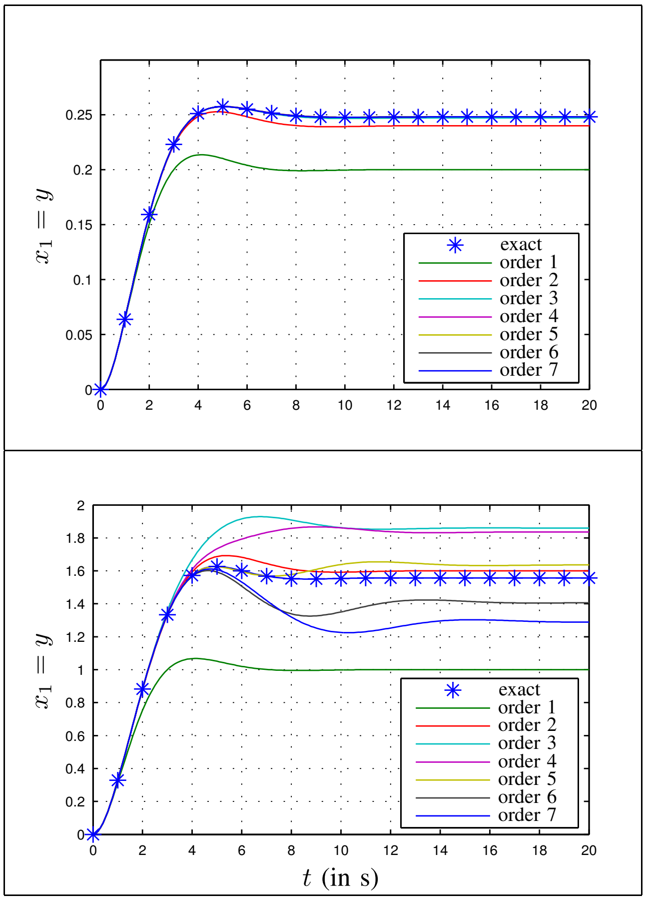

Fig. 2. Curve 1 (top): $u_1(t)=0.2$ and $u_2(t)=-0.2$. Curve 2 (bottom): $u_1(t)=1$ and $u_2(t)=-0.6$. These curves 1 and 2 are associated to the markers × and + in figure 1, respectively.

VI. Conclusion

Introducing an adequate functional setting, computable bounds of the convergence radius and of truncation errors of Volterra series expansions have been proposed for fading memory MI systems, analytic in state and affine in input. Results have been illustrated on one several example including one principal input and one auxiliary input.

Extensions to systems that are (in addition to the above assumptions) analytic in input and have nonzero initial conditions will be studied. Another task will consist of generalizing these results to some classes of infinite dimensional systems, such as boundary and distributed controlled PDE systems solved using Volterra series (see e.g. [17], [18]).

Appendix

A. Lemma 1 and proof

Lemma 1: Let $A(X)=\sum_{k=1}^{+\infty} a_k X^k$, $B(X)=\sum_{k=1}^{+\infty} b_k X^k$ be analytic functions at $X=0$ with non-negative coefficients. Let $\beta\in\mathbb{R}_+^*$. Define $F(X)=\dfrac{\beta+B(X)}{1-A(X)}$ and let $r\in\mathbb{R}_+^*\cup\{+\infty\}$ be the convergence radius of F at $x=0$. Then,

(i) At $x=0$, F is nonzero and analytic with nonnegative Taylor coefficients.

(ii) Equation $x\,F'(x)-F(x)=0$ has either one solution denoted σ (case 1) or zero solution (case 2), in $]0,r[$.

(iii) There exists a unique function $z\mapsto\Psi(z)$, analytic at $z=0$ such that $\Psi(z)=z\,F\big(\Psi(z)\big)$. Its convergence radius ρ_Ψ at $z=0$ is such that $\rho_\Psi=\rho^\star=\frac{\sigma}{F(\sigma)}$ in case 1 and $\rho_\Psi\ge\rho^\star=\lim_{x\to r^-}\frac{x}{F(x)}$ in case 2.

Proof: **(i):** If $A=0$, (i) is straightforward. Otherwise, A has at least one positive Taylor coefficients so that, for all $z\in\mathbb{C}$ such that $|z|<r$, $|A(z)|<A\big(|z|\big)<\lim_{x\to r^-}(x)\le 1$ and $F(z)=\big(\beta+B(z)\big)\sum_{n=0}^{+\infty}\big(A(z)\big)^n$, which proves (i). **(ii):** Define $H(x)=x\,F'(x)-F(x)$ for $x\in[0,r[$. If F is affine then $H(x)=-\beta$ so that $x\,F'(x)-F(x)=0$ has no solution. Otherwise, H is a strictly increasing function on $]0,r[$ from $H(0)<0$ to $\ell=\lim_{x\to r^-}H(x)\in\mathbb{R}\cup\{+\infty\}$ since for all $x\in]0,r[$, $H'(x)=x\,F''(x)>0$. Therefore, if $\ell>0$, then $x\,F'(x)-F(x)=0$ has a unique solution on $[0,r[$ (case 1), otherwise ($\ell\le 0$), it has no solution (case 2). **(iii):** In case 1, the hypotheses of the singular inversion theorem are met (see [19, prop. IV.5. and th. VI.6.]), and its application proves (iii). In case 2, (iii) is a direct consequence of the analytic inversion lemma (see [19, lemma 4.2.]). ■

B. Proof of theorem 5

Step 1: We prove by induction that, $\forall m\in\mathbb{N}_d^*$, h_m belongs to $\mathcal{V}_\lambda^{\overline{m}}$ and satisfies $\|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}}\le\psi_m$ where $\psi_{e_n}=\omega_n\ge\|h_{e_n}\|_{\mathcal{V}_\lambda^1}$, if $1\le n\le d$, and where, for all m such that $\overline{m}\ge 2$,

$$\psi_m \le \sum_{k=2}^{\overline{m}}\Bigg[\mathcal{P}_k\sum_{p\in\mathbb{M}_m^k}\prod_{i=1}^{k}\psi_{p_{*,i}}+\mathfrak{p}_k\sum_{\substack{q\in\mathbb{M}_m^k\\ q_{*,k}\in\{e_1,\dots,e_{d_\pi}\}}}(\omega^T\chi_\pi\ q_{*,k})\prod_{i=1}^{k-1}\psi_{q_{*,i}}$$
$$+\ \mathfrak{a}_k\sum_{\substack{q\in\mathbb{M}_m^k\\ q_{*,k}\in\{e_{d_\pi+1},\dots,e_d\}}}\prod_{i=1}^{k-1}\psi_{q_{*,i}}\Bigg],\qquad(20)$$

recalling that $q_{*,k}$ denotes the k-th column of matrix q.

The details of this step are similar to [16, (th. 2, step 1)] in which indexes are replaced with multiple indexes and Q_k is split as follows, using definition 8,

- if $q_{*,k}$ is one of the first d_π vectors of the canonical basis (associated to a principal input), then
$$Q_k(\dots,q_{*,k})=(\omega^T\,\chi_\pi\ q_{*,k})\,\widetilde{Q}_k^\pi(\dots,q_{*,k}),$$
- if $q_{*,k}$ is one of the last d_α vectors of the canonical basis (associated to a principal input), then
$$Q_k(\dots,q_{*,k})=\widetilde{Q}_k^\alpha(\dots,q_{*,k}).$$

Step 2: Consider the multi-variate formal series defined, for $X = (X_1, \dots, X_d)^T$, by $\Psi(X) = \sum_{m \in \mathbb{N}_d^\star} \psi_m X^m$ (generating function of ψ_m) where $X^m = X_1^{m_1} X_2^{m_2} \dots X_d^{m_d}$. The multivariate formal series

$$\mathcal{R}(X) = \sum_{k=2}^{+\infty} \Big[\mathcal{P}_k\big(\Psi(X)\big)^k + (\omega^T \chi_\pi X) \mathfrak{p}_k\big(\Psi(X)\big)^{k-1} + (\zeta^T X)\, \mathfrak{a}_k\big(\Psi(X)\big)^{k-1} \Big],$$

with $\zeta = (\underbrace{0, \dots, 0}_{d_\pi}, \underbrace{1, \dots, 1}_{d_\alpha})^T$, satisfies

$$\mathcal{R}(X) = \sum_{\left\{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} \geq 2}\right.} \psi_m X^m = \Psi(X) - \omega^T \chi_\pi X. \tag{21}$$

Let $\gamma \in \mathbb{R}^+$. Applying the change of variables $X \longrightarrow \widetilde{X}$ and denoting $\widetilde{\Psi}(\widetilde{X}) = \Psi(X)$ where $\widetilde{X} = \Big(\omega^T \chi_\pi X,\ \zeta^T X,\ X_2, \dots, X_{d_\pi},\ X_{d_\pi+2}, \dots, X_d\Big)$, equation (21) can be rewritten

$$\begin{aligned} & \widetilde{X}_1 \left[1 + \sum_{k=2}^{+\infty} \mathfrak{p}_k \Big(\widetilde{\Psi}(\widetilde{X})\Big)^{k-1} + \gamma \sum_{k=2}^{+\infty} \mathfrak{a}_k \Big(\widetilde{\Psi}(\widetilde{X})\Big)^{k-1} \right] \\ + \quad & \big(\widetilde{X}_2 - \gamma \widetilde{X}_1\big) \sum_{k=2}^{+\infty} \mathfrak{a}_k \Big(\widetilde{\Psi}(\widetilde{X})\Big)^{k-1} \\ = \quad & \widetilde{\Psi}(\widetilde{X}) \left[1 - \sum_{k=2}^{+\infty} \mathcal{P}_k \Big(\widetilde{\Psi}(\widetilde{X})\Big)^{k-1} \right]. \end{aligned}$$

In the quotient space $\mathbb{R}[[\widetilde{X}]]/(\widetilde{X}_2 - \gamma \widetilde{X}_1)$, this equation becomes $\widetilde{\Psi}(\widetilde{X}) = \widetilde{X}_1\ \mathcal{F}_{\lambda,\gamma}\big(\widetilde{\Psi}(\widetilde{X})\big)$.

From lemma 1 (iii), let $\widetilde{\Psi}_{\lambda,\gamma}$ be the unique function, analytic at $x = 0$, such that $\widetilde{\Psi}_{\lambda,\gamma}(x) = x\ \mathcal{F}_{\lambda,\gamma}\big(\widetilde{\Psi}_{\lambda,\gamma}(x)\big)$, with convergence radius $\rho_{\lambda,\gamma}$. Then, from definition (17),

$$\forall z \in V_{\lambda,\gamma}, \quad \Psi(z) = \widetilde{\Psi}_{\lambda,\gamma}(\omega^T \chi_\pi\, z), \tag{22}$$

where $V_{\lambda,\gamma}$ is defined by (17). Since $\|h_m\|_{\mathcal{V}_\lambda^{\overline{m}}} \leq \psi_m$ and from definition 7, Ψ is a majorizing function of φ_λ so that the Volterra series is convergent in $\mathcal{X}_\lambda$.

C. Proof of theorem 6

Let $u \in \Upsilon(V_\gamma)$ be such that $U < 1$. Define $a = \Big(\|u_1\|_{\mathcal{B}_\mathbb{R}(\lambda)}, \dots, \|u_d\|_{\mathcal{B}_\mathbb{R}(\lambda)}\Big)^T$. Then obviously, for all $n \in [1, d]_\mathbb{N}$, $a_n \leq \beta_n U$. Therefore, for all $N \in \mathbb{N}^*$,

$$\sum_{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} = N}} \psi_m\, a^m \ \leq\ U^N \sum_{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} = N}} \psi_m\, \beta^m. \tag{23}$$

Moreover, $\sum_{n=d_\pi+1}^{d} a_n = \gamma \rho^\star_{\lambda,\gamma}$ and $(\omega^T \chi_\pi \beta) = \rho^\star_{\lambda,\gamma}$ so that $\sum_{n=d_\pi+1}^{d} \beta_n = \gamma(\omega^T \chi_\pi \beta)$ and $(\omega^T \chi_\pi \beta)\, U < \rho^\star_{\lambda,\gamma}$. Therefore, from (22), function $U \mapsto \Psi(\beta\, U)$ is analytic on $]-1, 1[$ and $\Psi(\beta\, U) = \widetilde{\Psi}_{\lambda,\gamma}(\omega^T \chi_\pi \beta\, U) = \widetilde{\Psi}_{\lambda,\gamma}\big(\rho^\star_{\lambda,\gamma}\, U\big)$. Identifying the Taylor coefficients (w.r.t. U) of the left and right members of the latter equation shows that, for all positive integer N, $\sum_{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} = N}} \psi_m\, \beta^m = [\widetilde{\Psi}_{\lambda,\gamma}]_N\, (\rho^\star_{\lambda,\gamma})^N$, where $[\widetilde{\Psi}_{\lambda,\gamma}]_N$ denotes the Taylor coefficient of $\widetilde{\Psi}_{\lambda,\gamma}$ at order N. Replacing in (23), it comes $\sum_{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} = N}} \psi_m\, a^m \leq [\widetilde{\Psi}_{\lambda,\gamma}]_N\, (\rho^\star_{\lambda,\gamma}\, U)^N$ and it follows that $\|x - V_M x\|_{\mathcal{X}_\lambda} \leq \sum_{\substack{m \in \mathbb{N}_d^\star \\ \overline{m} \geq M+1}} \psi_m a^m \leq \sum_{n=M+1}^{+\infty} [\widetilde{\Psi}_{\lambda,\gamma}]_n (\rho^\star_{\lambda,\gamma} U)^n$.

In case 1 (see [16, proof of th.3], Cauchy estimates yield $[\widetilde{\Psi}_{\lambda,\gamma}]_n (\rho^\star_{\lambda,\gamma} U)^n \leq \sigma_{\lambda,\gamma} U^n$ so that $\|x - V_M x\|_{\mathcal{X}_\lambda} \leq \sigma_{\lambda,\gamma} \dfrac{U^{M+1}}{1-U} < +\infty$, which concludes the proof.

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