LYAPUNOV STABILITY ANALYSIS OF THE MOOG LADDER FILTER AND DISSIPATIVITY ASPECTS IN NUMERICAL SOLUTIONS

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ABSTRACT

This paper investigates the passivity of the Moog Ladder Filter and its simulation. First, the linearized system is analyzed. Results based on the energy stored in the capacitors lead to a stability domain which is available for time-varying control parameters meanwhile it is sub-optimal for time-invariant ones. A second storage function is proposed, from which the largest stability domain is recovered for a time-invariant Q-parameter. Sufficient conditions for stability are given. Second, the study is adapted to the nonlinear case by introducing a third storage function. Then, a simulation based on the standard bilinear transform is derived and the dissipativity of this numerical version is examined. Simulations show that passivity is not unconditionally guaranteed, but mostly fulfilled, and that typical behaviours of the Moog filter, including self-oscillations, are properly reproduced.

1. INTRODUCTION

The Moog Ladder Filter [1] is an analog audio device which is appreciated by many musicians because of its intuitive control, its sound singularity and its typical self-oscillating behaviour at high feedback-loop gains. This nonlinear analog circuit has been deeply studied and simulated using distinct methods [2, 3, 4, 5, 6, 7, 8].

Approaches based on energy and passivity considerations have proved to be relevant for simulating nonlinear systems, including applications to sound synthesis. This issue is of main importance for conservative systems whose simulation must neither diverge nor introduce parasitic dampings. These considerations have yet motivated works on e.g. nonlinear strings, plates and shells (see e.g. [9, 10, 11]). This approach is also worthwhile for dissipative systems, especially when they are able to reach conservative and self-oscillating behaviours. Thus, the filter of the EMS VCS3 synthesizer has been simulated using a decomposition of the circuit into modules which preserves passivity properties [12]. Moreover, these methods usually allow to derive simulations which are compatible with real-time computations usable by musicians.

In this paper, a study on the Moog filter passivity is performed from which a stability criterion is deduced according to the so-called Lyapunov stability analysis [13]. Lyapunov functions based on (a) the natural energy stored in the capacitors, (b) energies conveyed by eigenvectors of the linearized system and (c) some modified versions adapted to the nonlinear dynamics are considered. They allow to characterize stability domains as well as dissipated quantities. These features are applied to the discrete-time dynamics analysis of the bilinearly-transformed version of the system.

The outline of this paper is as follows. Section 2 recalls the circuit equations and provides a state-space representation of the system. Section 3 refreshes the definition of passivity, of Lyapunov functions as well as basic results on stability analysis. The stability analysis is performed, first, on the linearized version of the circuit in section 4, and then, on the original nonlinear circuit in section 5. Finally, in section 6, a dissipativity indicator especially designed for the bilinear transform is deduced, which allows to characterize the passivity preservation in simulated results.

2. CIRCUIT AND EQUATIONS

2.1. Circuit description

The Moog ladder filter is a circuit composed of (see Fig. 1 a-d): a driver, a cascade of four filters involving capacitors C, differential pairs of NPN-transistors and an additional feedback loop (detailed in (e)). Following [1], transistors are LM3046 or BC109a,b,c, po-

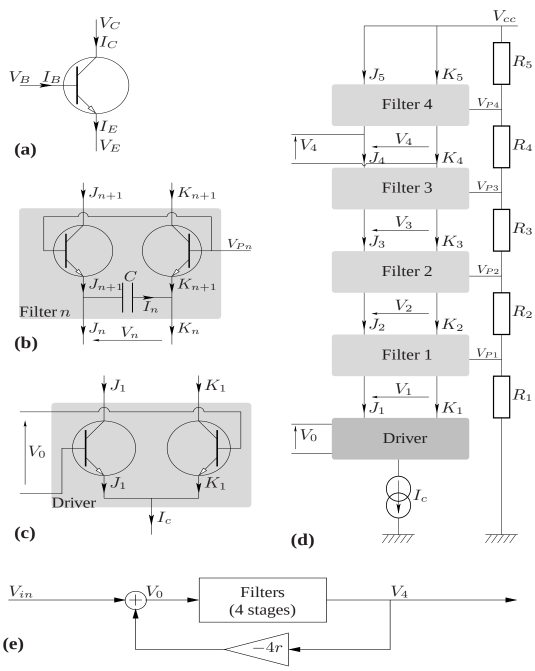

Figure 1: Circuits : (a) NPN transistor, (b) single-stage filter, (c) driver, (d) four-stages Moog ladder filter without loop and (e) Complete filter including a feedback-loop gain $-4r$.

larization voltages are $V_{P1} = 2.5\,\mathrm{V}$, $V_{P2} = 4.4\,\mathrm{V}$, $V_{P3} = 6.3\,\mathrm{V}$,

$V_{P4}=8.2$ V, $V_{cc}=15$V corresponding to $R_1=350$ Ohms, $R_2=R_3=R_4=150$ Ohms, $R_5=390$ Ohms. Capacitances are $C=27$ nF for the Moog Prodigy and $C=68$ nF for the MiniMoog synthesizer. Moreover, I_c is a voltage-controlled current which tunes a cut-off frequency and parameter r (in Fig. 1e) is a the voltage-controlled gain which monitors the resonance (no resonance if $r=0$, resonance with infinite Q-factor (self-oscillations) if $r=1$ [2]).

2.2. Circuit equations, energy and power balance

Denote q_n, I_n and V_n the charge, the current and the voltage of the capacitor in the n-th stage, respectively. The circuit equations are (see e.g. [8, § II] for a detailed derivation for transistor pairs)

$$\text{Capacitor law:} \qquad q_k = C\,V_k, \tag{1}$$

$$\text{Capacitor current:} \quad \frac{\mathrm{d}q_k}{\mathrm{d}t} = I_k, \tag{2}$$

$$\text{Transistor pair:} \qquad I_k = -\frac{I_c}{2}\tanh\frac{V_k}{2V_T} + \frac{I_c}{2}\tanh\frac{V_{k-1}}{2V_T}, \tag{3}$$

$$\text{Loop:} \qquad V_0 = V_{in} - 4rV_4. \tag{4}$$

where the thermal voltage is $V_T = k_b\,T/q \approx 25.85$ mV at temperature $T=300$ K ($k_b=1.38\,10^{-23}$ J/K is the Boltzmann constant, $q=1.6\,10^{-19}$ C is the electron charge).

The total energy which is stored in the capacitors is given by

$$E = \sum_{k=1}^{4} \frac{q_k^2}{2C}, \tag{5}$$

the time derivative of which yields the following power balance

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{k=1}^{4} \frac{q_k}{C}\frac{\mathrm{d}q_k}{\mathrm{d}t} = \sum_{k=1}^{4} V_k I_k, \tag{6}$$

which can be formulated as a function of $V_{0,\ldots,4}$ using (3).

2.3. Dimensionless version

Consider the dimensionless quantities

$$x_k=\frac{V_k}{V^\star}=\frac{q_k}{q^\star}, \quad i_k=\frac{I_k}{I^\star}, \quad u=\frac{V_{in}}{V^\star}, \quad e=\frac{1}{2}\,\frac{E}{E^\star},$$

with $V^\star=2V_T$, $q^\star=C\,V^\star$, $I^\star=\frac{I_c}{2}$ and $E^\star=\frac{(q^\star)^2}{2C}$. Then, (1) becomes trivial ($x_k=x_k$) since x_k characterizes both the voltage and the charge of capacitors. Moreover, (2-6) become, respectively, $\frac{1}{\omega}\frac{\mathrm{d}x_k}{\mathrm{d}t}=i_k$, $i_k=-\tanh x_k+\tanh x_{k-1}$, $x_0=x_4-4ru$,

$$e=\sum_{k=1}^{4}\frac{x_k^2}{2}, \quad \text{and} \quad \frac{\mathrm{d}e}{\mathrm{d}t}=\sum_{k=1}^{4} x_k\frac{\mathrm{d}x_k}{\mathrm{d}t}=\omega\sum_{k=1}^{4} x_k\,i_k, \tag{7}$$

which can be formulated as a function of $x_{1,\ldots,4}$ and u and where $\omega=I^\star/q^\star=I_c/(4C\,V_T)$ (in s^{-1}).

2.4. State-space representation and parameters

The dimensionless versions of (2-4) for $k=1,\ldots,4$ yield the state-space representation with state x and input u, given by

$$\frac{1}{\omega}\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,u) \text{ with } f(x,u)=\begin{bmatrix} -\tanh x_1+\tanh(u-4r\,x_4)\\ -\tanh x_2+\tanh x_1\\ -\tanh x_3+\tanh x_2\\ -\tanh x_4+\tanh x_3 \end{bmatrix}, \tag{8}$$

where the cutoff angular frequency $\omega=\frac{I_c}{4CV_T}$ has been extracted from f for sake of simplicity in the following derivations.

Remark 1. *In all the following equations (unless otherwise mentioned), the control parameters ω and r can depend on time and lie in $\omega\in\mathbb{R}_+^*$, $r\in[0,1]$ (the case $\omega=0$ which yields no dynamics $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ so that $x(t)=x(0)$ is discarded here). Note also that the signal which is usually used as the output of the filter corresponds to x_4.*

3. RECALLS ON PASSIVITY, STABILITY AND LYAPUNOV ANALYSIS

This section recalls some basic results about passivity, stability and Lyapunov analysis. The detailed theory can be found in [13, chapter 6].

Definition. Consider a system with state-space representation $\frac{\mathrm{d}x}{\mathrm{d}t}=f(x,u)$, output $y=\phi(x,u)$, where $f:\mathbb{R}^k\times\mathbb{R}^p\to\mathbb{R}^k$ is sufficiently regular (locally Lipschitz), $\phi:\mathbb{R}^k\times\mathbb{R}^p\to\mathbb{R}^p$ is continuous, $f(0,0)=0$ and $\phi(0,0)=0$. This system is said to be passive if there exists a continuously differentiable positive semidefinite function $\mathcal{V}(x)$ called the storage function such that $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{V}(x)\Big)\le y^Tu$. It is said to be *strictly passive* if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{V}(x)\Big)\le-\psi(x)+y^Tu, \tag{9}$$

for some positive definite function ψ.

Such a storage function $\mathcal{V}$ is called a *Lyapunov function*. It can be assimilated to an energy of the system and ψ to a dissipated power. One interest of the following approach is that it can be used for linear, nonlinear and possibly time-varying systems.

Links between passivity and stability. The passivity is a practical tool to examine some system stability aspects. It leans on the two following key points:

- when the excitation of the system stops ($u=0$), the positive storage function $\mathcal{V}$ stops increasing (passive system) or even decreases as long as $x\neq0$ (strictly passive system).
- If $\mathcal{V}\big(x(t)\big)$ is bounded, then x lives inside a closed bounded set: it cannot diverge. Moreover, since $\mathcal{V}$ is continuous and definite, if it decreases towards 0, then x also tends towards 0 (global asymptotic stability).

This very last case necessarily occurs for strictly passive systems with $u=0$ if $\mathcal{V}$ is radically bounded [13].

In short, this can be summarized as follows: *if a system with a suitable energy ($\mathcal{V}$) continuously dissipates some positive power, the system dynamics is bounded. Moreover, if function $\mathcal{V}$ is radically bounded, the dynamics eventually tends to a steady state ($x_0=0$ for $u=0$) which is stable.*

Passivity of systems in practice. Given a storage function $\mathcal{V}$, the passivity can be examined by deriving

$$\mathcal{P}(x,u)\overset{def.}{=}\Big(\partial_{(x^T)}\mathcal{V}(x)\Big)\,f(x,u)\,\Big(=\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}(x)\Big) \tag{10}$$

which represents a *power* ($\partial_{(x^T)}$ denotes the partial derivatives w.r.t. the row vector x^T). Indeed, if $\mathcal{P}(x,u)$ can be written as

$$\mathcal{P}(x,u)=-\psi(x)+\phi(x,u)^Tu \tag{11}$$

the passivity is obtained (with equality in (9)) w.r.t to the output $y=\phi(x,u)$.

4. LINEARIZED SYSTEM: STABILITY, ENERGY AND LYAPUNOV ANALYSIS

4.1. System equations, transfer matrix and pole analysis

4.1.1. Linear state-space equation

For $u = 0$, the unique equilibrium point of system (8) is zero. The linearized system around $(x, u) = (0, 0) \in \mathbb{R}^4 \times \mathbb{R}$ is given by

$$\frac{1}{\omega}\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + Bu \text{ with } A = \partial_x f(0,0) \text{ and } B = \partial_u f(0,0), \quad (12)$$

that is, $A = \begin{bmatrix} -1 & 0 & 0 & -4r \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

4.1.2. Transfer matrix and pole analysis

For time-invariant parameters ω and r, the input-to-state transfer matrix of (12) is

$$H(s) = F\left(\frac{s}{\omega}\right) \text{ where } F(\sigma) = \left(\sigma\, I_4 - A\right)^{-1} B$$

where I_4 denotes the 4×4 identity matrix, s denotes the Laplace variable and $\sigma = s/\omega$ is its dimensionless version.

The poles in the σ-complex plane are the roots of the characteristic polynomial

$$P_A(\sigma) = \det(\sigma\, I_4 - A) = \sigma^4 + 4\sigma^3 + 6\sigma^2 + 4\sigma + 1 + 4r.$$

For positive gains r, they are given by

$$\sigma_1 = -1 + \sqrt[4]{r} + i\sqrt[4]{r},\ \ \overline{\sigma_1},\ \ \sigma_2 = -1 - \sqrt[4]{r} + i\sqrt[4]{r} \text{ and } \overline{\sigma_2}. \quad (13)$$

Their real parts are all strictly negative for gains $r < 1$. This condition characterizes the strict stability domain.

Remark 2. *As $s = \omega\sigma$ and σ-poles do not depend on ω, changing ω modifies the cutoff frequency of the filter. But, it does not modify the quality factor of the resonance which is exclusively tuned by r. This is an appreciated particularity of the Moog filter since it makes its control easier (see [2]).*

4.1.3. Conclusion on stability

For time-invariant parameters, the pole analysis reveals that the linearized system is strictly stable (poles have all a strictly negative real part) if the positive gain r satisfies $r < 1$. The case $r = 1$ corresponds to the limit of stability. The linear filter becomes unstable for $r > 1$. The stability domain does not depend on ω. Finally, for constant parameters, the strict stability maximal domain is

$$(\omega, r) \in \mathcal{D}_{max} = \mathbb{R}_+^* \times [0, 1[. \quad (14)$$

4.2. Passivity analysis based on the natural circuit energy

This section tries to restore the result (14) with a Lyapunov approach, from the passivity analysis of the linearized system. Parameters can be time-varying.

4.2.1. Passivity analysis

The energy of a physical system is a natural candidate Lyapunov function. For the (dimensionless) Moog filter, it corresponds to the sum of the energies stored in the four capacitors, given by (7), which can be rewritten as $e = \mathcal{V}(x)$ with

$$\mathcal{V}(x) = \sum_{k=1}^{4} V(x_k), \quad (15)$$

where V is the energy of one capacitor

$$V(x) = \frac{1}{2}x^2, \quad (16)$$

(see § 2.3 for the conversion into dimensional physical quantities).

Using the matrix formulation $\mathcal{V}(x) = \frac{1}{2}x^T x$, (10) leads to $\mathcal{P}(x,u) = \omega\left(\partial_x \mathcal{V}(x)\right)^T\left(Ax + Bu\right) = \omega\left(-\widetilde{\psi}(x) + \widetilde{\phi}(x,u)^T u\right)$ where

$$\widetilde{\psi}(x) = -x^T A x \quad \text{and} \quad \widetilde{\phi}(x,u) = Bx = \omega x_1. \quad (17)$$

The strict passivity is obtained w.r.t. the output $y = x_1$, if $\widetilde{\psi}$ is positive definite. This is the case if and only if the symmetric matrix $Q = Q^t = -\frac{1}{2}\left(A + A^T\right)$ from which $\widetilde{\psi}(x) = x^T Q x$ can also be defined is positive definite. This condition is equivalent to

$$\forall k \in [1,4]_{\mathbb{N}}, \quad \det \begin{bmatrix} Q_{1,1} & \dots & Q_{k,1} \\ \vdots & & \vdots \\ Q_{k,1} & \dots & Q_{k,k} \end{bmatrix} \stackrel{def}{=} d_k(Q) > 0. \quad (18)$$

The matrix Q is given by

$$Q = Q^t = -\frac{1}{2}(A + A^T) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 2r \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 2r & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

The sub-determinants $d_k(Q)$ are given by

$$d_1(Q) = 1, \quad d_2 = \frac{3}{4}, \quad d_3 = \frac{1}{2} \quad \text{and} \quad d_4 = \frac{5}{16} + \frac{1}{2}r - 3r^2.$$

They are all strictly positive iff $d_4 > 0$, that is, $-\frac{1}{4} < r < \frac{5}{12}$.

4.2.2. Conclusion

The Lyapunov analysis based on the natural energy does not restore the maximal stability domain $\mathcal{D}_{max}$ given in (14) but only the subset $\mathcal{D}_{natural}$ described by

$$(\omega, r) \in \mathcal{D}_{natural} = \mathbb{R}_+^* \times [0, \frac{5}{12}[\subset \mathcal{D}_{max}. \quad (19)$$

This result which can appear contradictory at first sight is actually a quite well-known feature of the Lyapunov stability analysis, as stated below.

Remark 3. *The Lyapunov stability analysis gives a sufficient condition for stability. But, it does not allow to conclude that a system is unstable if the condition is not fulfilled. Moreover, it does not guide the user in choosing the Lyapunov function (whether optimal or not). These difficulties have motivated many works to derive a candidate Lyapunov function for a given system, witnessed by a large recent bibliography on this topic. This is precisely why using the natural energy of physical systems as a Lyapunov function is usually appreciated. Unfortunately, this function is not optimal here.*

At this step, it is useless to study the stability of the nonlinear system and its self-oscillating limit which is precisely known to be reached at $r = 1$: a Lyapunov function allowing to restore the stability domain $\mathcal{D}_{max}$ must be investigated first.

4.3. Recovery of the maximal stability domain

As there is no general constructive method to design adapted Lyapunov functions, an inductive (but constructive) method is introduced below, which consists in writing the power balance in a basis of eigenvectors so that the dissipativity is straightforwardly related to the real part of eigenvalues, that is, the poles of the transfer function.

First, the method which yields a parametrized family of candidate Lyapunov functions to test is described. Second, the method is applied to the Moog system: as for the pole analysis, the obtained result restores the upper bound for time-invariant gains r.

4.3.1. Description of the proposed method

In this section, parameter r (but not ω) is assumed to be time-invariant. The method is described by the 4 following steps:

Step 1. Write the diagonal version of the system in a basis of eigenvectors $v_1, \overline{v_1}, v_2, \overline{v_2}$ associated with eigenvalues $\Sigma = (\sigma_1, \overline{\sigma_1}, \sigma_2, \overline{\sigma_2})$ defined in (13), that is,

$$\frac{1}{\omega}\frac{\mathrm{d}x_\star}{\mathrm{d}t} = A_\star\, x_\star + B_\star u$$

with $x_\star = P^{-1}x$, $A_\star = P^{-1}AP = \mathrm{diag}(\Sigma)$, $B_\star = P^{-1}B$ and

$$P = \left[v_1, \overline{v_1}, v_2, \overline{v_2}\right]. \tag{20}$$

Step 2. Build a power balance from this state equation as follows

$$x_\star^H\, W_\star \left(\frac{1}{\omega}\frac{\mathrm{d}x_\star}{\mathrm{d}t}\right) = x_\star^H\,(W_\star\, A_\star)\, x_\star + x_\star^H\,(W_\star\, B_\star)\, u,$$

where $W_\star = \mathrm{diag}(w_\star)$ with $w_\star \in (\mathbb{R}_+^*)^4$ can be any positive definite "diagonal weight matrix" ($x_\star^H = \overline{x_\star}^T$ denotes the hermitian, i.e. the transposed conjugate version of $x_\star$).

Step 3. The average of the latter equation and its hermitian version yields (recalling that u is real valued)

$$\frac{1}{\omega}\frac{\mathrm{d}e_\star}{\mathrm{d}t} = -\widetilde{\psi}_\star(x_\star) + \widetilde{\phi}_\star(x_\star)^T u, \tag{21}$$

where $e_\star = \frac{1}{2}x_\star^H W_\star x_\star$, $\phi_\star(x_\star) = \Re\mathrm{e}(B_\star^H\, W_\star\, x_\star)$ and $\psi_\star(x_\star) = x_\star^H\, Q_\star\, x_\star$ with $Q_\star = -\Re\mathrm{e}(W_\star\, A_\star)$, so that the two following key points are fulfilled:

(i) $e_\star$ is a positive definite function of $x_\star$ (since $W_\star$ is positive definite);

(ii) $Q_\star$ is positive definite if and only if all the eigenvalues σ (the poles of the transfer matrix) have a strictly negative real part.

Step 4. Write these results in the original state basis: (10) is recovered with $e_\star = \mathcal{V}(x)$, $\frac{1}{\omega}\,\mathcal{P}(x,u) = -\widetilde{\psi}(x) + \widetilde{\phi}(x)^T u$, and

$$\mathcal{V}(x) = \frac{1}{2}x^t W x \quad \text{with } W = W^H = P^{-H} W_\star P^{-1}, \tag{22}$$

$$\widetilde{\psi}(x) = x^t Q x \quad \text{with } Q = Q^H = -P^{-H}\Re\mathrm{e}(W_\star\, A_\star) P^{-1}, \tag{23}$$

$$\widetilde{\phi}(x) = L\, x \quad \text{with } L = B^T\, P^{-H} W_\star P^{-1} = B^T\, W, \tag{24}$$

for any $W_\star = \mathrm{diag}(w_\star) > 0$ and where $P^{-H} = (P^{-1})^H$.

This method builds a family of "candidate" Lyapunov functions parametrized by a definite positive diagonal weight matrix $W_\star = \mathrm{diag}(w_\star)$.

Remark 4 (Time-invariant parameter r)**.** *The validity of (21) is conditioned by the fact that $w_\star$ does not depend on time and that of step 4 by the fact that P does not depends on time. Actually, P (related to A) depends on r (but not ω) so that the method gives "candidate" Lyapunov functions for time-invariant r and possibly time-varying ω).*

4.3.2. Application

In step 1, the computation of eigenvectors leads to

$$v_1 = \left[\ \sqrt[4]{r}^3,\ +\frac{1-i}{2}\sqrt{r},\ -\frac{i}{2}\sqrt[4]{r},\ -\frac{1+i}{4}\ \right]^T, \quad \text{for } \sigma = \sigma_1,$$

$$v_2 = \left[\ \sqrt[4]{r}^3,\ -\frac{1+i}{2}\sqrt{r},\ +\frac{i}{2}\sqrt[4]{r},\ +\frac{1-i}{4}\ \right]^T, \quad \text{for } \sigma = \sigma_2,$$

and $P^{-1} = \left[v_1, \overline{v_1}, v_2, \overline{v_2}\right]^{-1}$ is given by

$$P^{-1} = \begin{bmatrix} 1 & 1+i & i & -1+i \\ 1 & 1-i & -i & -1-i \\ 1 & -1+i & -i & 1+i \\ 1 & -1-i & i & 1-i \end{bmatrix} \left(\mathrm{diag}\left[4\sqrt[4]{r}^3,\ 4\sqrt{r},\ 2\sqrt[4]{r},\ 2\ \right]\right)^{-1}.$$

Then, Choosing an uniform weight $w_\star = 4\sqrt{r}^3\,[1,1,1,1]^T$, the results of step 4 leads to (22-24) with

$$\begin{aligned} W &= \mathrm{diag}(w), \quad \text{with } w = \left[1, 2\sqrt{r}, 4r, 8\sqrt{r}^3\right]^T, \\ Q = Q^T &= \begin{bmatrix} 1 & -\sqrt{r} & 0 & 2r \\ * & 2\sqrt{r} & -2r & 0 \\ * & * & 4r & -4\sqrt{r}^3 \\ * & * & * & 8\sqrt{r}^3 \end{bmatrix}, \\ L &= [1,0,0,0]. \end{aligned}$$

The sub-determinants (18) are given by

$$d_1(Q) = 1, \qquad d_2(Q) = (2-\sqrt{r})\sqrt{r},$$
$$d_3(Q) = 8\sqrt{r}^3(1-\sqrt{r}) \quad \text{and} \quad d_4(Q) = 64r^3(1-\sqrt{r})^2.$$

They are all strictly positive for $r \in]0,1[$.

Hence, the maximal domain is restored, except the special case $r = 0$. Actually, in this case, the power balance based on eigenvectors is degenerated, since eigenvectors becomes all collinear and, eventually, do no take account of all capacitors anymore.

4.4. Results summary: Passivity and asymptotic stability of the linearized Moog filter

The previous studies (§ 4.2-4.3) allow to state the following result.

The linearized version of the Moog Ladder Filter (§ 4.1.1) is strictly passive and its equilibrium point $(x,u) = (0,0)$ is asymptotically stable under one of the following conditions:

case a: for all time-varying parameters (ω, r) lying in

$$\mathcal{D}_{natural} = \mathbb{R}_+^* \times [0, \frac{5}{12}[.$$

case b: for all time-varying parameter ω and time-invariant parameter r lying in

$$\mathcal{D}_{max} = \mathbb{R}_+^* \times [0, 1[.$$

Moreover, the dynamics of the system fulfills strictly passive power balances w.r.t. the output $y = \omega\widetilde{\phi}(x,u)$, based on adapted Lyapunov functions given by, for cases $\eta = a$ and $\eta = b$,

$$\mathcal{V}^{\eta}_{\ell in}(x) = w_\eta^T \, \underline{V}_{\ell in}(x) \tag{25}$$

with $\underline{V}_{\ell in}(x) = \Big[V_{\ell in}(x_1), \ldots, V_{\ell in}(x_4)\Big]^T$, and $V_{\ell in}(x_k) = \dfrac{x_k^2}{2}$,

and (10) with $\frac{1}{\omega}\mathcal{P}_\eta(x,u) = -\widetilde{\psi}_\eta(x) + \widetilde{\phi}_\eta(x,u)^T u$

$$\widetilde{\psi}_\eta(x) = x^T Q_\eta x \quad \text{with } Q_\eta = -\frac{1}{2}\Big(W_\eta A + A^T W_\eta\Big)$$
$$\widetilde{\phi}_\eta(x,u) = L_\eta x \quad \text{with } L_\eta = B^T W_\eta$$

and where $\widetilde{\psi}_\eta$ is positive definite choosing $W_\eta = \mathrm{diag}(w_\eta)$ with

$$w_a = [1,1,1,1]^T, \quad w_b = \big[1, 2\sqrt{r}, 4r, 8\sqrt{r}^3\big]^T, \text{ (if } r(t) = r(0) = r).$$

Remark 5 (Local dissipativity). *Although the case $\eta = b$ gives a strong result only for a constant parameter r, the study developed in § 4.3 allows to state a weaker result, including for time-varying parameters lying in $\mathcal{D}_{max}$.*
Let $r \in [0,1[$ be a fixed value. Consider $\ell = a$ if $r = 0$, $\ell \in \{a,b\}$ if $0 < r < 5/12$ and $\ell = b$ if $5/12 \le r < 1$. Moreover, denote $(\tilde{\omega}, \tilde{r})$ the time-varying parameters which monitor the linearized filter, lying in $\mathcal{D}_{max}$. If $\tilde{r}(t) = r$ at a time t, then the power balance (10) is fulfilled at time t.

5. NONLINEAR SYSTEM ANALYSIS

In this section, the passivity and the Lyapunov stability are examined for the nonlinear system. The study is based on a storage function which allows to obtain a formulation similar to that of the linearized system and to take benefit from the results stated in section 4. The derivations are presented in three steps. First, a remarkable identity on the feedback loop is exhibited. Second, the identity is used to reformulate the state-space equation (8) in a way similar to (12). Third, the storage function, the passivity and the Lyapunov analysis are presented.

5.1. Step 1: remarkable identity on the feedback loop

From $\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$, we get $\tanh(u - 4rx_4) = -\tanh(4rx_4) + \frac{\left(1-\tanh^2(4rx_4)\right)\tanh u}{1 - \tanh(4rx_4)\tanh u}$ which rewrites

$$\tanh(u - 4rx_4) = -4\,\rho_r(x_4)\,\tanh(x_4) + \beta(rx_4, u)\,u, \tag{26}$$

where functions ρ_r and β are positive regular functions.
The function $\rho_r : \mathbb{R} \longrightarrow I_r = \rho_r\langle\mathbb{R}\rangle$ is defined by

$$\rho_r(x_4) = \frac{\tanh(4r\,x_4)}{4\tanh x_4}, \ \text{ if } x_4 \neq 0, \text{ and } \rho_r(0) = r, \tag{27}$$

where $I_0 = \{0\}$, $I_r = [r, \frac{1}{4}[$ if $0 < r < \frac{1}{4}$, $I_{\frac{1}{4}} = \{\frac{1}{4}\}$ and $I_r =]\frac{1}{4}, r]$ otherwise (see Fig. 2 for an illustration). Hence, a property is that, for all $x_4 \in \mathbb{R}$, if $r \in [0,1[$ then $\rho_r(x_4) \in [0,1[$ as well.
The function $\beta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined by

$$\beta(z,u) = \frac{1 - \tanh^2 z}{1 - \tanh u \,\tanh z}\,\mu(u), \tag{28}$$

$$\text{with } \mu(u) = \frac{\tanh u}{u}, \quad \text{if } u \neq 0, \quad \text{and } \mu(0) = 1. \tag{29}$$

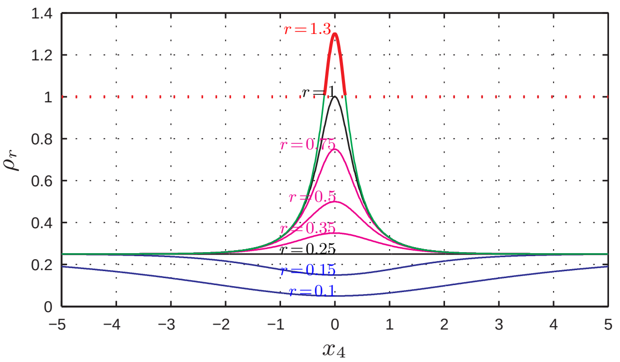

Figure 2: *Function $x_4 \mapsto \rho_r(x_4)$ for several positive values of r. This function plays the same role (a feedback gain) as r in the linearized system, for the nonlinear system. For $r = 1.3$, the red part exceeds the stability limit gain value 1.*

Note that in (28), the first factor is positive, finite and regular (but not bounded), and that μ is positive, finite, regular and lower than 1 (see Fig. 3 for an illustration), so that function β is well-posed.

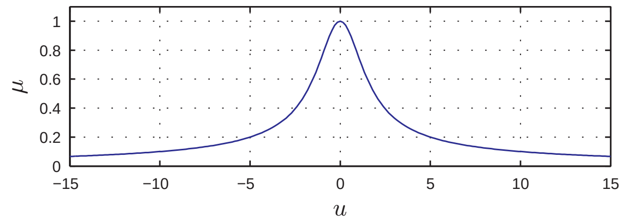

Figure 3: *Function $u \mapsto \mu(u)$ defined in (29).*

5.2. Step 2: State-space formulation similar to (12)

Using identity (26) in (8) and introducing

$$\Theta(x) = [\tanh x_1, \ \tanh x_2, \ \tanh x_3, \ \tanh x_4\,]^T, \tag{30}$$

lead to the state-space equation

$$\frac{1}{\omega}\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{A}\,\Theta(x) + \mathcal{B}\,u \tag{31}$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are functions of, respectively, $\rho_r(x_4)$ and $\beta(rx_4, u)$, which are given by

$$\mathcal{A} = \begin{bmatrix} -1 & 0 & 0 & -4\rho_r(x_4) \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} \beta(x,u) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{32}$$

Remark 6 (Gains). *$\rho_r(x_4)$ can be interpreted as a feedback gain and $L(rx_4, u)$ as the input gain.*

5.3. Step 3: Storage function, passivity and Lyapunov analysis

Let $w \in (\mathbb{R}_+^*)^4$, $W = \mathrm{diag}(w)$ and consider the storage function $\mathcal{V}_{n\ell}$ defined as (25) where $V_{\ell in}$ is replaced by (see Fig. 4)

$$\forall x_k \in \mathbb{R}, \quad V_{n\ell}(x_k) = \ln\cosh(x_k). \tag{33}$$

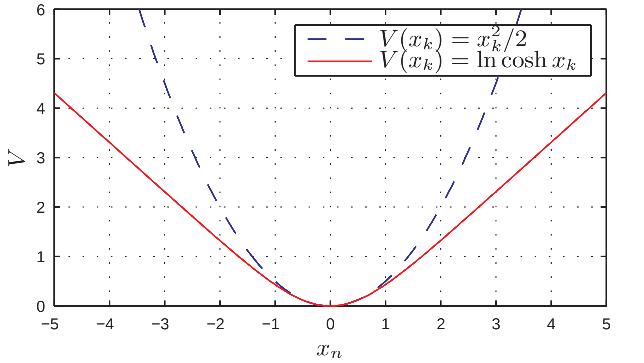

Figure 4: *Storage functions: $V_{\ell in}$ (- -) defined by (16) is used for the linearized system analysis and $V_{n\ell}$ (-) (33) for the nonlinear system analysis.*

Remark 7 (Function $V_{n\ell}$). *The storage function $V_{n\ell}$ defined by (33) does not longer correspond to the energy stored in a capacitor, except for small signals since $\ln\cosh x \underset{0}{\sim} \frac{1}{2}x^2$ (see also Fig. 4).*

Then, following (31), and as $V'_{n\ell}(x_k) = \tanh x_k$,

$$\begin{aligned}\frac{1}{\omega}\frac{\mathrm{d}\mathcal{V}_{n\ell}(x)}{\mathrm{d}t} &= \left(\Theta(x)^T W\right) f(x,u)\\ &= \Theta(x)^T W\,\mathcal{A}\Theta(x) + \Theta(x)^T W\mathcal{B}u,\end{aligned}$$

so that $\frac{\mathcal{P}_{n\ell}(x)}{\omega} = -\widetilde{\psi}_{n\ell}(x) + \widetilde{\phi}_{n\ell}(x,u)\,u$ where

$$\begin{aligned}\widetilde{\psi}_{n\ell}(x) &= \Theta(x)^T Q_{n\ell}(\rho_r)\Theta(x) \quad \text{with } Q_{n\ell} = -\frac{1}{2}(W\mathcal{A}+\mathcal{A}W),\\ \widetilde{\phi}_{\eta}(x,u) &= L_{n\ell}(x,u)\Theta(x) \quad \text{with } L_\eta = \mathcal{B}^T W.\end{aligned}$$

As $\widetilde{\psi}_{n\ell}$ is a quadratic form w.r.t. $\Theta(x)$, it is positive definite iff $Q_{n\ell}$ is a positive definite matrix as well. This property is achieved as in section 4 for $w = w_a$ since $\mathcal{A}$ is the same matrix as A in which $r \in [0, 5/12[$ has been replaced by $\rho_r(x_4) \in [0, 5/12[$.

As a consequence, the passivity w.r.t $y = \omega\widetilde{\phi}_a$ and the Lyapunov stability are proved for any time-varying parameter lying in $\mathcal{D}_{natural}$ (§ 4.4-case $\eta = a$). Nevertheless, the results of (§ 4.4-case $\eta = b$) cannot be used here, but only the remark 5, since $\rho_r(x_4)$ varies with x_4 (even for fixed r).

5.3.1. *Conclusion*

Loosing the result of section 4.4 (case $\eta = b$) for the nonlinear system appears awkward. In practice, this is compensated by the fact that the feedback-loop gain ρ_r ($\forall r > 0$) becomes stabilizing again for large x_4 as soon as it falls under the critical value 1 (or $5/12$ for $\mathcal{V}_{n\ell}$ with the weight $w = [1,1,1,1]$), as stated in the remark below. Finding a constant weight w restoring $\mathcal{D}_{max}$ for the linear case would solve also the problem for the nonlinear case.

Remark 8 (Stability limit). *The stability limit is reached when the gain ρ_r equals to 1, that is, only at $x_4 = 0$ in the case where $r = 1$. But, the more x_4 deviates from 0, the more $\rho_r(x_4)$ decreases (see Fig 2) and reinforces the stabilization. In this case, $x = 0$ is a limit stable equilibrium point for $u = 0$. If $r \geq 1$, the system is locally unstable on $x_4 \in \{x_4 \,|\, \rho_r(x_4) > 1\} = I_r$ (see the red part in Fig. 2), but locally stabilized on $\mathbb{R} \setminus I_r$. Since $x = 0$ is the only equilibrium point for $u = 0$, the system can become self-oscillating.*

6. NUMERICAL SCHEME AND DISSIPATIVE BEHAVIOUR

In signal processing, the bilinear transform is an extensively used numerical scheme. One reason is that it preserves the stability domain for time-invariant linear filters and usually lead to expected behaviours also for some time-varying and nonlinear cases. The question addressed here is to estimate how this numerical scheme, used here as in [2, 3], is able to fulfill a power balance close to a discrete-time (DT) version of (9-11).

For sake of conciseness, the notations used in the following part are $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,u,\alpha)$ where α denote the (possibly time-varying) parameters $\alpha = (\omega, r)$ and $x(n)$ denotes the variable at time $t = n\tau$ (rather than $x(n\tau)$) for the sampling frequency $F_s = 1/\tau$.

6.1. Bilinear transform

Applying the bilinear transform to (8) leads to

$$\begin{aligned}x(n{+}1) - x(n) = \frac{\tau}{2}\Big[& f\big(x(n{+}1), u(n{+}1), \alpha(n{+}1)\big)\\ &+ f\big(x(n), u(n), \alpha(n)\big)\Big]. \quad (34)\end{aligned}$$

The simulation of the dynamics is processed by computing

$$x(n+1) = x(n) + \delta(n), \qquad (35)$$

where $\delta(n)$ is governed by (34) in which $x(n+1)$ is replaced by (35). The computation of $\delta(n)$ is processed either by using a Newton-Raphson algorithm [14] or by approximating the solution by that of the first order Taylor expansion (w.r.t. $\delta(n)$) of its governing equation, that is,

$$\begin{aligned}\delta(n) = \frac{\tau}{2}&\Big[I_4 - \frac{\tau}{2}\partial_{(x^T)} f\big(x(n), u(n{+}1), \alpha(n{+}1)\big)\Big]^{-1}\\ &\times\Big[f\big(x(n), u(n), \alpha(n)\big) + f\big(x(n), u(n{+}1), \alpha(n{+}1)\big)\Big]. \quad (36)\end{aligned}$$

In practice, the latter solution is accurate if the cutoff frequency is sufficiently low ($\omega/(2\pi) \ll F_s/2$). In this case, the computation cost can still be reduced without deteriorating the result by replacing the last factor of (36) by $f\Big(x(n), \frac{u(n+1)+u(n)}{2}, \frac{\alpha(n+1)+\alpha(n)}{2}\Big)$.

6.2. Discrete-time power balance and dissipated contribution

The passivity is examined for the storage function $\frac{\mathrm{d}\mathcal{V}_{n\ell}(x)}{\mathrm{d}t} = w^T \frac{\mathrm{d}\underline{V}_{n\ell}(x)}{\mathrm{d}t}$ with $\underline{V}_{n\ell}(x) = [V_{n\ell}(x_1), \ldots, V_{n\ell}(x_4)]^T$. Following (34-35), a DT version of $\frac{\mathrm{d}V_{n\ell}(x_k)}{\mathrm{d}t} = V'_{n\ell}(x_k)\frac{\mathrm{d}x_k}{\mathrm{d}t}$ is

$$\frac{V_{n\ell}\big(x_k(n{+}1)\big) - V_{n\ell}\big(x_k(n)\big)}{\tau} = \Delta V\big(x_k(n), \delta_k(n)\big)\,\frac{\delta_k(n)}{\tau},$$

$$\Delta V(\xi, \delta) = \frac{V_{n\ell}(\xi{+}\delta) - V_{n\ell}(\xi)}{\delta} \quad \Big(\xrightarrow[\delta\to 0]{} V'_{n\ell}(\xi) = \tanh\xi\Big),$$

where $\delta(n)$ is computed using (34-35).

To characterize the dissipativity in the sense of this DT power balance, the associated function ψ^d must be identified. This can be done remarking that, according to (10-11), functions $\psi(x,\alpha) = w^T \underline{\psi}(x)$ and $\phi(x,u,\alpha) = w^T \underline{\phi}(x,u,\alpha)$ are here formally computed with $\psi_k(x_k,\alpha) = -V'_{n\ell}(x_k) f(x,0,\alpha)$ and $\phi_k(x,u,\alpha) = \Big[V'_{n\ell}(x_k) f_k(x_k,u,\alpha) + \psi(x_k,\alpha)\Big]/u$. The looked-for DT versions are then given by replacing $V'_{n\ell}$ by δV and occurrences of f by their corresponding averages at samples n and $n+1$. The DT dissipated contribution ψ^d computed in this way is given by

$$\psi^d\Big(x(n),\delta(n);\alpha(n),\alpha(n+1)\Big) = -\underline{\Delta V}\big(x(n),\delta(n)\big)^T \times \mathrm{diag}(w) \times \frac{f\big(x(n),0,\alpha(n)\big) + f\big(x(n)+\delta(n),0,\alpha(n+1)\big)}{2}. \quad (37)$$

with $\underline{\Delta V}\big(x,\delta\big) = \big[\Delta V(x_1,\delta_1),\ldots,\Delta V(x_4,\delta_4)\big]^T$ and makes sense. It yields a DT Lyapunov principle for $u=0$ if $\psi^d \geq 0$: $V_{n\ell}\big(x_k(n+1)\big) - V_{n\ell}\big(x_k(n)\big) = -\tau\,\psi^d\Big(x(n),\delta(n);\alpha(n),\alpha(n+1)\Big)$. The positivity domain of ψ_d is not straightforward to exhibit even for constant parameters (ω, r): in this case, ψ can be written as a quadratic form w.r.t. $\theta(n)+\theta(n+1)$ but with a weight matrix which depends on the time, as stated in the following remark.

Remark 9. *For constant parameters, ψ is a quadratic function w.r.t. $T(\xi,\delta) = \big(\tanh(\xi+\delta) + \tanh\xi\big)/2$. This is obtained rewriting $\Delta V\big(\xi,\delta\big) = F(\xi,\delta)\,T(\xi,\delta)$ where the introduced function F can be interpreted as a correction factor due to the time-discretization. This factor is proved to be larger than 1 and such that $F(\xi,\delta) \underset{\delta\to 0}{\longrightarrow} 1$.*

6.3. Simulations and results

Simulations presented below are performed using (36) with $F_s = 48$kHz. The input is a linear sweep $u(t) = a\sin\phi(t)$ with $\phi(t) = 2\pi\big(f^- t + \frac{f^+ - f^-}{2t^\star} t^2\big)$ starting with frequency $f^- = 50$ Hz and ending with $f^+ = 2$ kHz at $t^\star = 0.1$ s. Three amplitudes are tested: $a = 0.01$ (very linear limit, except if the resonance is high), $a = 1$ (medium nonlinear dynamics) and $a = 5$ (highly nonlinear dynamics). The cutoff frequency is $f_c = 1$ kHz.

For $r \leq 5/12$ (low resonance and limit of the proved passivity for the linear approximation of the filter, see § 4.2.2), the DT passivity is unconditionally satisfied ($\psi^d \geq 0$) for the indicator ψ^d built with the weight $w = w_a$ (energy stored in capacitors).

For $5/12 < r < 1$, this is no longer the case but the DT passivity is still mostly fulfilled for the indicator ψ^d built with the weight $w_b(r)$. The dissipativity violation can be appreciated in Fig. 5 for $r = 0.7$ (high resonance) and $r = 1.1$ (non asymptotically stable domain).

Note that, even in the latter case ($r > 1$), x_4 does not diverge. For the very low amplitude ($a = 0.01$), a self-oscillation appears, as for the analog circuit. For larger amplitudes, the filter is driven by the input so that no self-oscillation appear.

Hence, the bilinear transform has not proved to guarantee the passivity. However, this numerical study shows its relevance and its ability to capture some of the characteristic and expected behaviours of the Moog filter.

7. CONCLUSIONS AND PERSPECTIVES

In this paper, the passivity analysis of the Moog Ladder Filter circuit has been examined. Three families of candidate Lyapunov functions have been proposed for the linearized and the nonlinear versions of the system: in the linear case, the natural energy stored in capacitors and an adapted weighted sum of the energies conveyed by eigenvectors and, in the nonlinear case, an adaptation of elementary storage functions. The first one guarantees the passivity for any time-varying parameters on a restricted domain. The second one recovers the optimal stability domain for time-invariant feedback-loop gains in the linear case. The third one generalizes these results to the nonlinear case in an exact way for the first one, but it only gives some clues for the characterization of the optimal domain. Finally, these results have allowed to derive an dissipativity indicator for discrete-time simulations based on the bilinear transform.

The analysis of simulations reveals that the bilinear transform does not guarantee the dissipativity of the nonlinear filter if gains r are larger than $5/12$ (whether time-varying or not). However, even in this case, the dissipativity condition stays mostly fulfilled and simulations show that the bilinear transform generates the known characteristic behaviours of the Moog filter.

As a consequence, the main perspective of this study consists in deriving refined storage functions and a specific numerical scheme that (both) guarantee a dicrete-time passivity. Conservative schemes (which preserve the energy of conservative systems and Hamiltonian systems) based on variational approaches have been developed [15] and are still an active field of research. They can reveal to be relevant also when they are applied to lossy versions of originally non-lossy problems (see e.g. [9, Apdx. A1]). Another perspective is concerned with the derivation of explicit schemes preserving the passivity. Finally, a deeper study on the time variation effects of parameter $r > 5/12$ should be carried on.

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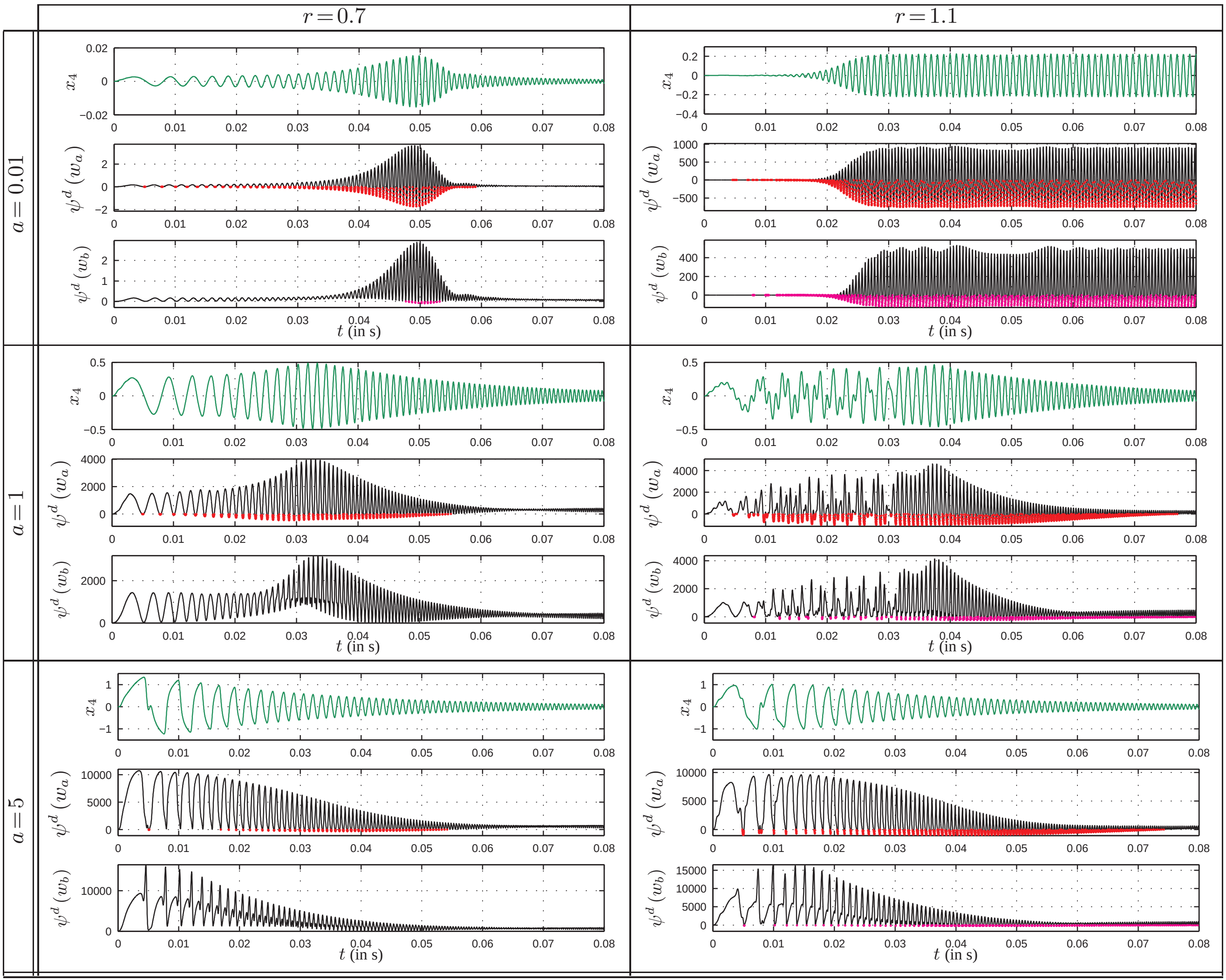

Figure 5: *Compilation of simulations of x_4 (filter output) and of ψ^d with the weights w_a and $w_b(r)$. The dissipativity violation corresponds to $\psi^d < 0$ (associated samples are marked in red for w_a and in magenta for w_b). As expected, there are very few marked samples for $r = 0.7$. The time at which $f_{sweep}(t) = f_c$ is $t = 48.7$ ms.*

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