On damping models preserving the eigenfunctions of conservative systems: a port-Hamiltonian perspective ⋆

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Abstract: In this paper, a special class of damping model is introduced for second order dynamical systems. This class is built so as to leave the eigenfunctions invariant, while modifying the dynamics: for mechanical systems, well-known examples are the standard fluid and structural dampings.
In the finite-dimensional case, the so-called Caughey series are a general extension of these standard damping models; the damping matrix can be expressed as a polynomial of a matrix, which depends on the mass and stiffness matrices. Damping is ensured whatever the eigenvalues of the conservative problem if and only if the polynomial is positive for positive scalar values. This can be recast in the port-Hamiltonian framework by introducing a port variable corresponding to internal energy dissipation (resistive element). Moreover, this formalism naturally allows to cope with systems including gyroscopic effects (gyrators).
In the infinite-dimensional case, the previous polynomial class can be extended to rational functions and more general functions of operators (instead of matrices), once the appropriate functional framework has been defined. In this case, the resistive element is modelled by a given static operator, such as an elliptic PDE. These results are illustrated on several PDE examples: the Webster horn equation, the Bernoulli beam equation; the damping models under consideration are fluid, structural, rational and generalized *fractional* Laplacian or bi-Laplacian.

Keywords: energy storage, port-Hamiltonian systems, eigenfunctions, damping, Caughey series, partial differential equations, fractional Laplacian

1. INTRODUCTION

In this paper, the idea is to find and even parametrize damping models of discrete systems (or ODEs) and continuous systems (or PDEs), which leave the eigenvectors or eigenfunctions unaffected by the damping: only the eigenvalues are shifted. To this end, in 1896, Lord Rayleigh introduced damping models named after him, which are nothing but a first order polynomial in both the mass and stiffness matrix. But the pioneering works by Caughey in 1960, shortly followed by Caughey and O'Kelly in 1965 showed a more general result: it is the structure of the *commutant* of the two matrices, or two operators, which play a central role in the theory. Hence, not only polynomials of this compound matrix prove admissible, but also series of this matrix, hence the famous *Caughey series*.

The main idea of the work is to take advantage of port-Hamiltonian framework, see e.g. van der Schaft and Maschke (2004), and Duindam et al. (2009) for a guided tour, to treat this question, and see how Caughey polynomials, rational functions, or even more general functions can fit into it. The extension to systems of PDEs will be looked at, with naive examples as well as new worked-out examples.

The outline of the paper is as follows: in § 2, a general second order n-d.o.f mechanical system is studied, with a quite general damping matrix, we first put it into the port-Hamiltonian framework, in order to introduce both skew-symmetric and symmetric structural matrices J and R. We first concentrate on the properties for the G-part of the damping, responsible of the so-called *gyroscopic* effects. Then, after an adequate change of coordinate has been performed, we give the desirable properties for the C-part of damping, in order to follow the so-called *Basile* hypothesis, that is the damped system still has classical normal modes. The nice result by Caughey, back to 1960, made more precise in 1965, is fully recalled. Different classes of solutions are examined : polynomial, rational functions and more general functions of matrices, provided a positivity constraint is fulfilled.

In § 3, we turn to the PDE case, and try to follow the same approach as before: it turns out that the commutation of operators (including the boundary conditions in their domain) happens to be the key point of the result, as first mentionned by the pioneering work by Caughey & O'Kelly

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in 1965 : thus, we extend Rayleigh damping models to Caughey type operators, which amount to polynomials, rational functions or even more general functions (such as *fractional* powers) of a compound operator : this can be treated seriously e.g. in the case of unbounded operators with compact resolvent, that are coercive and self-adjoint; a nice exaple of those is provided by the coupling with an elliptic PDE. In this section, a focus is made on worked-out examples such as the Webster wave equation (that allow for space-varying coefficients), and also Bernoulli beam model.

Finally in § 4, we give many questions that this preliminary work on damping has raised, many interesting perspectives are listed, and some ideas towards solutions are also provided, giving as broad as possible a perspective on this difficult subject.

2. FINITE-DIMENSIONAL SYSTEMS: EQUIVALENT DESCRIPTIONS AND INTRODUCTION OF DAMPING MODELS

2.1 Harmonic oscillator

We start with the port-Hamiltonian formulation of the n-d.o.f. finite dimensional harmonic oscillator. Dynamic equation is usually written in the form:

$$M\ddot{x} + (C+G)\dot{x} + Kx = 0\,, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ and $M = M^T > 0$, $K = K^T \geq 0$ and the damping matrix is decomposed into its symmetric part $C = C^T$, and its skew-symmetric part $G = -G^T$. By using as state variables the energy variables (i.e. the position and the momentum) and defining the Hamiltonian H_0 as the total energy of the system, *i.e.*:

$$X := \begin{bmatrix} q = x, \\ p = M\dot{x} \end{bmatrix} \quad \text{and} \quad H_0(X) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} q^T K\, q\,;$$

it is possible to rewrite (1) in the form of a port-Hamiltonian system:

$$\frac{d}{dt}X = \begin{bmatrix} 0 & I \\ -I & -(G+C) \end{bmatrix} \partial_X H_0(X) = (J-R)\,\partial_X H_0(X)\,.$$

where $\partial_X H_0(X) = \begin{bmatrix} Kq = Kx \\ M^{-1}p = \dot{x} = v \end{bmatrix}$, and:

$$J := \begin{bmatrix} 0 & I \\ -I & -G \end{bmatrix} \quad \text{and} \quad R := \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$$

J is full rank $2n$ and skew-symmetric , whereas R is symmetric positive (when $C = C^T \geq 0$), with rank equal to at most n, thus not positive definite.

2.2 About the G matrix

This matrix is often not considered in modelling processes of damping, why? Because in fact it has no damping effect, of course, since the simple computation shows that, whatever the value of G (skew-symmetric), when $C = 0$ (which is equivalent to $R = 0$), the system is conservative: $\frac{d}{dt}H_0(X(t)) = 0$.

Hence the question arises [1] : is it a naive generalizations by mathematicians, or does there exist mechanical examples of systems with such a matrix? Of course the dimension must be $n \geq 2$, otherwise $g = 0$. Let $n = 3$, and consider the Coriolis force with rotational speed $\omega = (p, q, r)^T$; then the classical term $\omega \wedge \dot{x}$ is nothing but $G_\omega\, \dot{x}$, with

$$G_\omega := \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}.$$

Finally, it is quite an easy exercise to prove how a simple change of co-ordinates enables to reduce the conservative part to the canonical symplectic structure (i.e. with G=0, to put it shortly): it amounts to describe the dynamics in the rotating axe system; hence, from now on, $G = 0$ is taken for granted.

2.3 Structures for the C matrix: results, discussion and examples

In Caughey (1960), setting $N := M^{1/2}$, $\tilde{C} := N^{-1}CN^{-1}$ and $\tilde{K} := N^{-1}KN^{-1}$ (which are still symmetric positive matrices), a *sufficient* condition is found for our problem, namely that $\tilde{C}$ be a series in $\tilde{K}$. Finally, taking advantage of the well-known Cayley-Hamilton theorem in finite dimension, it is found to be equivalent that $\tilde{C}$ be a polynomial in $\tilde{K}$.

Note that the more general result, which is a *necessary and sufficient* condition proved in Caughey and O'Kelly (1965), is

$$[\tilde{C}, \tilde{K}] = 0$$

where $[A, B] := AB - BA$ is the commutant; we then recover the previous sufficient condition as a special case [2] .

In order to use the degrees of freedom given by Caughey, some attempts have been made in e.g. Adhikari (2006), but the right change of variable is not performed ($M^{-1}K$ is never a symmetric matrix, hence the results of this paper are highly questionable, at least from a mathematical point of view), even if some results seem interesting for applications.

2.3.1 The polynomial case Suppose we want to put the $\tilde{C} := \sum_{l=0}^{n-1} q_l\, \tilde{K}^l$ damping model into the port-Hamiltonian framework, first we must reinterpret this relation as

$$C := \sum_{l=0}^{n-1} q_l\, KM^{-1}K \cdots M^{-1}K\,,$$

each term having l occurrences of K and $l-1$ of M^{-1}, second we can put it in the dissipative framework used, e.g. in Villegas et al. (2006), by introducing external effort e_p and flow variables f_p, which are linked by a closure relation $e_p = S\, f_p$, with $S = S^T \geq 0$. Let us focus on the

[1] The first author would like to thank Prof. J. Kergomard for fuitful discussion on this subject, first giving the right name to this term, then giving the example of Coriolis effect in solid mechanics.

[2] In fact, working this more general condition out shows that our problem could have been not well-posed from the very beginning: we should have been asking for invariance of *eigen-subspaces*, rather than eigenvectors; this better way of rephrasing the problem will have to be handled in future works.

first order developpment, that is $C := q_0 M + q_1 K$ with $q_0, q_1 \geq 0$. Let

$$G_p := \begin{bmatrix} 0 & 0 \\ M^{1/2} & K^{1/2} \end{bmatrix} \quad \text{and} \quad S := \mathrm{diag}(q_0 I, q_1 I)\,.$$

System (1) can now be written as:

$$\begin{bmatrix} f \\ f_p \end{bmatrix} = \begin{bmatrix} J & G_p \\ -G_p^T & 0 \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} \quad \text{and} \quad e_p = S\, f_p\,.$$

The feedback form does correspond to the following dynamics:

$$\dot{X} = (J - G_p S G_p^T)\, \partial_X H_0(X)\,.$$

For higher order developpments, such as $C := q_0 M + q_1 K + q_2 K M^{-1} K$, builind up G_p and S proves the same, but presents the disadvantage of making explicit use of M^{-1}, which would be preferable not to compute in many circumstances, at least from a numerical point of view [3]

2.3.2. Other interesting parametrizations? For short, it is a good idea to write $\tilde{C} := f(\tilde{K})$, where function f is well defined in the *cone of symmetric positive matrices*, which readily amounts to diagonalize the transformation in an orthonormal basis, and apply $c_i := f(k_i)$ on each coordinate, with $k_i \geq 0$. Now a condition for damping is that $f(\mathbb{R}^+) \subset \mathbb{R}^+$, so as to ensure $\tilde{C} := f(\tilde{K}) \geq 0$, hence $C \geq 0$. As special cases, not using the Cayley-Hamilton theorem from the beginning, it can be interesting to make a distinction between:

(1) polynomials, defined explicitely by: $\tilde{C} := Q(\tilde{K})$, such as Rayleigh damping when $\deg(Q) = 1$, see § 2.3.1,
(2) rational functions, which can also be defined implictely by: $P(\tilde{K})\, \tilde{C} := Q(\tilde{K})$,
(3) irrational functions, such as $\tilde{C} = \tilde{K}^\alpha$, see e.g. appendix A.

It seems that the second subclass, when coupled to the global dynamics, has a link with *Differential Algebraic Equations*, see e.g. Kunkel and Mehrmann (2006), and also so-called *Descriptor Systems* in the field of automatic control.

It is interesting to try to put also the last two parametrizations into a pHs-like framework with dissipation, but so far it seems an open question. Note that this question, which could sound formal in finite dimension, should be answered in a nice way before passing to the infinite-dimensional setting, where no Cayley-Hamilton result of any kind can be accounted for.

3. INFINITE-DIMENSIONAL SYSTEMS: THEORY AND EXAMPLES

We now turn to PDE models, or continuous systems. It is indeed the underlying geometric structure of PDEs which must be considered and put forward in our studies, as in Arnold (2004). We first give an example of the gyroscopic effect for PDEs, and then turn to structuring the C operatorm, which is the main purpose of this section. We start in § 3.2.1 with the results of introducing Rayleigh damping into the constant coefficient Euler-Bernoulli model of the beam, and show from an engineering point of view the interest of tuning the two parameters of the first order polynomial. Preliminary theoretical results are given in § 3.2.3, and are fully illustrated in two different new cases in § 3.2.4: rational damping for Webster horn equation with space-varying coefficients, and irrational damping such as fractional Laplacian or bi-Laplacian.

3.1 Gyroscopic effects in infinite dimension?

Here is a simple example [4]: an ideal and incompressible fluid is governed by $\frac{d}{dt}\boldsymbol{v} = -(\boldsymbol{v}.\boldsymbol{grad})\boldsymbol{v} - \frac{1}{\rho_0}\boldsymbol{grad}(p)$ and $div(\boldsymbol{v}) = 0$. After some computations, we find that $\forall \phi,\ \psi \in H_0^1(\Omega)$,

$$\int_\Omega \psi\, \boldsymbol{v}.\boldsymbol{grad}\varphi \ \mathrm{d}V = -\int_\Omega \left(\varphi\, \boldsymbol{v}.\boldsymbol{grad}\psi + div(\boldsymbol{v})\, \varphi\psi\right)\ \mathrm{d}V\,.$$

Hence, thanks to the divergence-free condition, the operator $G : \varphi \mapsto \boldsymbol{v}.\boldsymbol{grad}\varphi$ is skew-symmetric w. r. t. $L^2(\Omega)$:

$$(\psi, \boldsymbol{v}.\boldsymbol{grad}\varphi)_{L^2(\Omega)} = -(\varphi, \boldsymbol{v}.\boldsymbol{grad}\psi)_{L^2(\Omega)}\,;$$

this non-uniform convection term definitely plays the role of a gyroscopic term in infinite dimension.

3.2 Structuring the $\mathcal{C}$ operators: preliminary results, discussion and examples

3.2.1. Rayleigh damping for the Euler-Bernoulli beam: sound examples of damped bars made of metal, glass or wood Consider a dimensional version of the Euler-Bernoulli's beam model (see Graff (1975)), excited by the force f at $z = 0$ and with free end at $z = L$, which includes a fluid and a structural damping. For a constant cross-section and a homogeneous material, it corresponds to the following equations (see Hélie and Matignon (2001))

$$YI\partial_z^4 u + \rho S\left[a + b\, \partial_z^4\right] \partial_t u(t,z) + \rho S\, \partial_t^2 u = 0 \quad (2)$$

$$\partial_z^2 u(t,0) = \partial_z^2 u(t,L) = 0 \quad \text{(no momentum)} \quad (3)$$

$$\partial_z^3 u(t,0) = f(t)\ \text{(force) and}\ \partial_z^3 u(t,L) = 0 \text{(no force)}. \quad (4)$$

In this model, ρ and Y are the density and the Young's modulus of the material, respectively, and $I = \frac{wh^3}{12}$ is the geometrical momentum of the bar (w is the width and h the height). Positive coefficients a and b quantify the effect of the fluid and the structural dampings, respectively.

Simulations based on a modal decomposition has been proposed in Hélie and Matignon (2001) for realistic sound synthesis purposes, with the following sensible physical values: $L = 0.5\,\mathrm{m}$ (bar length), $w = 0.05\,\mathrm{m}$ (width), $h = 0.0117\,\mathrm{m}$ (height), $Y = 2.13\ 10^{10}\,\mathrm{Pa}$ (Young's modulus) $\rho = 1015\,\mathrm{Kg.m}^{-3}$ (purple wood density). When no damping is present, the first and last considered modes correspond to frequencies $f_1 = 220\,\mathrm{Hz}$ and $f_{12} = 15190\,\mathrm{Hz}$, respectively.

As the damping coefficients are unknown, several physical orders of magnitude are presented: three sounds are synthetised and their respective spectrograms are presented in figures 1. Qualitatively, these examples show that b is representative of wooden bar sounds (marimba), whereas a is more representative of metallic bar sounds (vibraphone).

[3] Another choice is possible, which circumvents this difficulty, with $S = \mathrm{diag}(M, K)$ and G parametrized by $\sqrt{q_0}$, $\sqrt{q_1}$, but this somewhat nicer decomposition does not generalize easily to PDEs.

[4] The first author would like to thank Prof. L. Jezequel for fuitful discussion on this subject, and mentioning the example of fluid mechanics in a duct with convection.

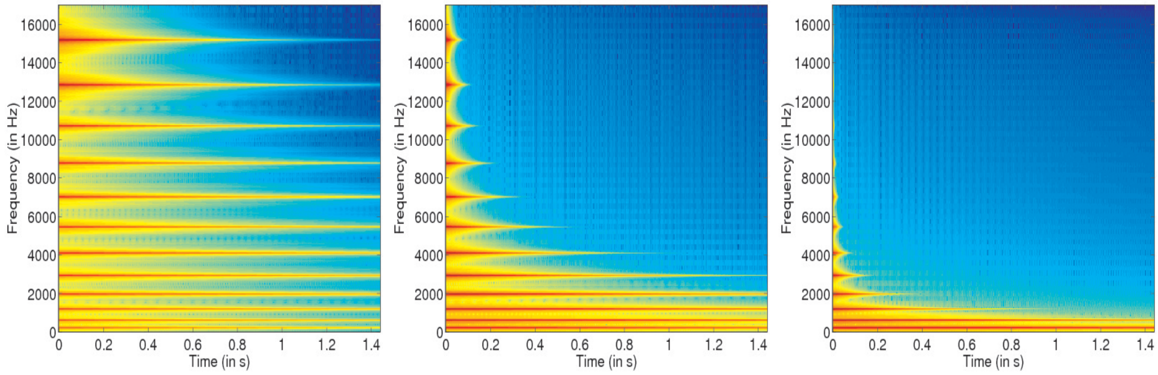

Fig. 1. Rayleigh type dampings: spectrogram of $\partial_t^2 u(t,L)$. (l): $a = 4e-2$ and $b = 3e-9$ (SI), sounds like a metallic bar, (c): $a = 2e-2$ and $b = 5e-8$ (SI), sounds like a glass bar, (r): $a = 1e-2$ and $b = 5e-7$ (SI) sounds like a wooden bar.

It can be heard that both dampings give rise to different audible behaviours and provide a large set of sounds close to percussive bar sounds.

The spectrograms show how, on a practical example, such damping models can be used to improve the sound synthesis realism: both a and b are required.

For Rayleigh damping on conservative PDEs, analyzed in e.g. Jacob et al. (2008), a port-Hamiltonian formulation is available in e.g. Villegas et al. (2006); we recall it here, for sake of clarity. In a simplified way, denoting $v = \partial_t u$, the dynamics now reads $\partial_{tt}^2 u + y(v) + \partial_{z^4}^4 u = 0$, with damping term $y(v) := q_0 v + q_1 \partial_{z^4}^4 v$. Classically, $q = \partial_{z^2}^2 u$ and $p = \partial_t u$, with Hamiltonian $H_0 = \frac{1}{2}\int_0^L (q^2 + p^2)\,\mathrm{d}z$. We can compute the variational derivatives $\delta_q H_0 = q$ and $\delta_p H_0 = p$, and check

$$\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 & \partial_{z^2}^2 \\ -\partial_{z^2}^2 & 0 \end{bmatrix}\begin{bmatrix} \delta_q H_0 \\ \delta_p H_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{bmatrix}\begin{bmatrix} \delta_q H_0 \\ \delta_p H_0 \end{bmatrix};$$

which has the desired $(\mathcal{J} - \mathcal{R})$ form, with $\mathcal{J}$ skew-symmetric and $\mathcal{R}$ symmetric. In order to parametrize $\mathcal{R} = \mathcal{G}\mathcal{S}\mathcal{G}^*$, we define next

$$\mathcal{G} := \begin{bmatrix} 0 & 0 \\ 1 & \partial_{z^2}^2 \end{bmatrix} \quad \text{and} \quad \mathcal{S} := \mathrm{diag}(q_0 I, q_1 I)\,,$$

which helps describe the whole system, using the extended efforts and flows :

$$\begin{bmatrix} f \\ f_p \end{bmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{bmatrix}\begin{bmatrix} e \\ e_p \end{bmatrix} \quad \text{and} \quad e_p = \mathcal{S}\, f_p\,.$$

The feedback form which is obtained corresponds indeed to the damped dynamics:

$$\dot{X} = (\mathcal{J} - \mathcal{G}\mathcal{S}\mathcal{G}^*)\,\partial_X H_0(X)\,.$$

3.2.2. Navier-Stokes equation perfectly fits into the dissipative pHs framework Following van der Schaft and Maschke (2001), we consider an irrotational and isentropic fluid, in a bounded domain $\Omega \subset \mathbb{R}^3$. Using standard notations, the dynamical equations of the fluid can be written as:

$$\frac{d}{dt}\rho = -div(\rho\,\boldsymbol{v}) \tag{5}$$

$$\frac{d}{dt}\boldsymbol{v} = -(\boldsymbol{v}.\boldsymbol{grad})\boldsymbol{v} - \frac{1}{\rho}\boldsymbol{grad}p + \frac{1}{Re}\boldsymbol{\Delta v}\,. \tag{6}$$

where pressure p is derivable from a potential energy density $U(\rho)$, as $p = \rho^2\frac{\partial U}{\partial\rho}$. Re is Reynold's number. Hence, with Hamiltonian

$$H_0 := \int_\Omega \left(\frac{1}{2}\rho\boldsymbol{v}.\boldsymbol{v} + \rho U(\rho)\right)\,\mathrm{d}V\,,$$

we first compute the variational derivatives $\delta_{\boldsymbol{v}} H_0 = \rho\,\boldsymbol{v}$ and $\delta_\rho H_0 = \frac{1}{2}\boldsymbol{v}.\boldsymbol{v} + h(\rho)$ (with $h(\rho) := U(\rho) + \rho\,\frac{\partial U}{\partial\rho}$ being the enthalpy), and then rewrite equations (5)-(6) as [5] :

$$\frac{d}{dt}\begin{bmatrix} \rho \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 0 & -div \\ -\boldsymbol{grad} & 0 \end{bmatrix}\begin{bmatrix} \delta_\rho H_0 \\ \delta_{\boldsymbol{v}} H_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{bmatrix}\begin{bmatrix} \delta_\rho H_0 \\ \delta_{\boldsymbol{v}} H_0 \end{bmatrix};$$

with $\mathcal{C} = -\frac{1}{Re}\boldsymbol{\Delta}$. It has the desired $(\mathcal{J} - \mathcal{R})$ form: $\mathcal{J}$ is skew-symmetric, since the formal adjoint of div is $-\boldsymbol{grad}$, and $\mathcal{R}$ is symmetric and positive, since $-\boldsymbol{\Delta}$ is. More important, the parametrization $\mathcal{R} = \mathcal{G}\mathcal{S}\mathcal{G}^*$ is very easily found to be [6] :

$$\mathcal{G} := \begin{bmatrix} 0 \\ \boldsymbol{grad} \end{bmatrix}, \quad \mathcal{G}^* = [0\ \ -div]\,, \quad \text{and} \quad \mathcal{S} := \frac{1}{Re}\,I\,.$$

3.2.3. More general damping models: preliminary theoretical results Once again, for second order in time models, a *sufficient* condition proved in Caughey and O'Kelly (1965), is given by commutation of the reduced operators, (including their domain). Note that all the operators involved ($\mathcal{M}$, $\mathcal{C}$, $\mathcal{K}$) are supposed to be self-adjoints, $\mathcal{M}$ being coercive, and $\mathcal{K}$ positive. Letting $\mathcal{N} := \mathcal{M}^{1/2}$, we define $\tilde{\mathcal{C}} := \mathcal{N}^{-1}\mathcal{C}\mathcal{N}^{-1}$ and $\tilde{\mathcal{K}} := \mathcal{N}^{-1}\mathcal{K}\mathcal{N}^{-1}$. Thus the condition reads (including the domains of these reduced operators):

$$[\tilde{\mathcal{C}}, \tilde{\mathcal{K}}] = 0$$

The paper gives many counter-examples, either due to the structure of the operators, or their domains; an example is also provided.

As special cases, it proves very interesting to make a distinction between:

(1) polynomials, defined explicitely by: $\tilde{\mathcal{C}} := Q(\tilde{\mathcal{K}})$, such as Rayleigh damping when $\deg(Q) = 1$, see § 3.2.1,
(2) rational functions, which can also be defined implictely by: $P(\tilde{\mathcal{K}})\,\tilde{\mathcal{C}} := Q(\tilde{\mathcal{K}})$,
(3) irrational functions, such as $\tilde{\mathcal{C}} = \tilde{\mathcal{K}}^\alpha$.

[5] The identity $(\boldsymbol{v}.\boldsymbol{grad})\boldsymbol{v} = \boldsymbol{grad}(\frac{1}{2}\boldsymbol{v}.\boldsymbol{v})$ is true, since $\boldsymbol{rot}(\boldsymbol{v}) = \boldsymbol{0}$.
[6] The identity $\boldsymbol{\Delta v} = \boldsymbol{grad}(div(\boldsymbol{v}))$ is true, since $-\boldsymbol{rot}(\boldsymbol{rot}(\boldsymbol{v}) = \boldsymbol{0}$.

3.2.4. More general damping models: new worked-out examples We now explore two examples as illustrations of the more general classes, as suggested by the above theory: when more functions than just polynomials are allowed, it gives rise to a wider variety of behaviours.

3.2.4.1. Rational damping for Webster horn equation This model is a wave equation, which has coefficients $S(z)$ variable in space, it is put in conservative form. A first order rational function of $\tilde{\mathcal{K}} = -\partial_z(S(z)\partial_z)$ is being used for $\tilde{\mathcal{C}}$: let $v := \partial_t u$ and define $y(v)$ as solution to the *elliptic* PDE:

$$p_0 y - p_1 \partial_z(S\partial_z y) = q_0 v - q_1 \partial_z(S\partial_z v)\,,$$

where $p_1, q_1 \geq 0$, and $p_0, q_0 > 0$. With boundary conditions, this problem is well-posed, thanks to Lax-Milgram theorem. The positivity condition, $\int_0^L y(z)\, v(z)\,\mathrm{d}z \geq 0$, can be cheked thanks to a spectral mapping theorem and $f(\mathbb{R}^+) \subset \mathbb{R}^+$ when $f(z) := \frac{q_0+q_1\,z}{p_0+p_1\,z}$; but still, more precise results can be proved. Using $d := q_1 p_0 - p_1 q_0 \neq 0$, two cases may occur:

(1) when $d > 0$, then $v := \frac{p_1}{q_1} y + w$ implies $\frac{d}{q_1} y = q_0 w + q_1 \tilde{\mathcal{K}} w$, just like Rayleigh damping, which garantees $(y, v) = \frac{p_1}{q_1}\|y\|^2 + (y, w) \geq 0$, since $\frac{d}{q_1}(y, w) = q_0\|w\|^2 + q_1(\tilde{\mathcal{K}} w, w) \geq 0$.
(2) when $d < 0$, then $y := \frac{q_1}{p_1} v + z$ implies $p_0 z + p_1 \tilde{\mathcal{K}} z = -\frac{d}{p_1} v$, which guarantees $(y, v) = \frac{q_1}{p_1}\|v\|^2 + (z, v) \geq 0$, since $-\frac{d}{p_1}(z, v) = p_0\|z\|^2 + q_1(\tilde{\mathcal{K}} z, z) \geq 0$.

Recasting these two models in a port-Hamiltonian setting does not prove straightforward, even using van der Schaft and Maschke (2004).

3.2.4.2. Fractional Laplacian or bi-Laplacian: irrational damping models Also of interest is the case of fractional Laplacian or bi-Laplacian (still with ideal boundary conditions), see Hansen (2000) and references therein for this specific type of fractional damping model: $\tilde{\mathcal{C}} = \tilde{\mathcal{K}}^\alpha$. We refer to Matignon (2009) for careful definitions of such *non rational* functions of operators [7].

The main idea behing this somewhat quite general damping model, is to see the root locus it gives rises to: explicit analytical computations can be carried out on $y(v) := q_0 v + (-\Delta)^\alpha v$, but we briefly show the root locus as a function of the α parameter on figures 2 and 3:

- for $0 < \alpha < 0.5$, the dynamical system is of hyperbolic type, the roots are located on a parabolic branch $\Im m(s) \propto (-\Re e(s))^\nu$ with $\nu = \frac{1}{2\alpha} > 1$,
- for $\alpha = 0.5$, the asymptote is a straight line ($\nu = 1$),
- for $0.5 < \alpha$, the dynamical system is of parabolic or diffusive type, the roots are eventually located on $\mathbb{R}^-$, with only finitely many damped oscillating roots (located on a circle when $\alpha = 1$, Rayleigh damping).

[7] A key point is the *compactness* property: when it is present, this property enables to write down things into series instead of finite sums (with the celebrated sine, cosine or *Fourier series* on $L^2(I)$, where I is a bounded interval), and this applies both to bounded and unbounded operators in fact. When it is not present, general integrals instead of series have to be considered: the celebrated *Fourier transform* on $L^2(\mathbb{R})$ is also recalled.

4. CONCLUSION AND PERSPECTIVES

We have looked for a structuration of the damping models which preserve the classical normal modes of the undamped structure, the Basile hypothesis. For discrete systems, or ODEs, the Caughey series has been put in the formalism of port-Hamiltonian system, the different cases have been examined and illustrated polynomial, rational function and even more general functions satisfying the positivity constraint. For continuous systems, or PDEs, the general ideas behind Caughey series have also been put into the port-Hamiltonian setting, at least formally, and a few interesting examples have been treated.

Moreover, many points are to be looked at carefully, in the continuation of this preliminary work on structuration of damping, such as:

- how to formulate the implicit cases in a pHs setting?
- how to use these models for the purpose of identification of damping parameters?
- use some operational calculus on non-normal operators? Think of Riesz basis as directely related to Hilbert basis and then use this as a foundation for operational calculus: is that too naive an idea? In which case, how does the positivity constraint translates? Into a *positive real* condition, such as $\Re e(f(s)) \geq 0$ for $\Re e(s) > 0$?

And, last but not least, an objection could very much be raised before going on: what is the real interest, and on what physical ground, do we look for normal modes in damped structures? Different answers are possible: one could argue that eigenvalues are affected at the first order when a slight damping is applied, whereas eigenvectors or eigenfunctions are only moved up to the second order of the damping parameter. Moreover, for many physical problems, refined damping models are not available. For instance, in applications such as in section 3.2.1 (see e.g. Causse et al. (2011)), an engineering approach is often used, which consists in computing the modal decomposition of the conservative problem and introducing, a posteriori, a specific damping for the dynamics of each mode according to some heuristics. Damping models that preserve the eigen-functions of the conservative problem exactly address this issues but, in an intrinsic way, that is, without having to derive the eigenstructure. This gives both a formal framework and define an equivalence class of damped models.

Finally, pHs formalism proves most useful when modelling damping for PDEs: when non ideal boundary conditions are present, not simply Dirichlet or Neumann, such as Robin type or more general impedance boundary conditions, there is a need to clarify the underlying structure, which could very much be given, almost for free, by the port variables in the pH framework: this is, at the best of our knowledge, one of the most important reason to turn to pHs for PDEs in order to build and define coherent damping models.

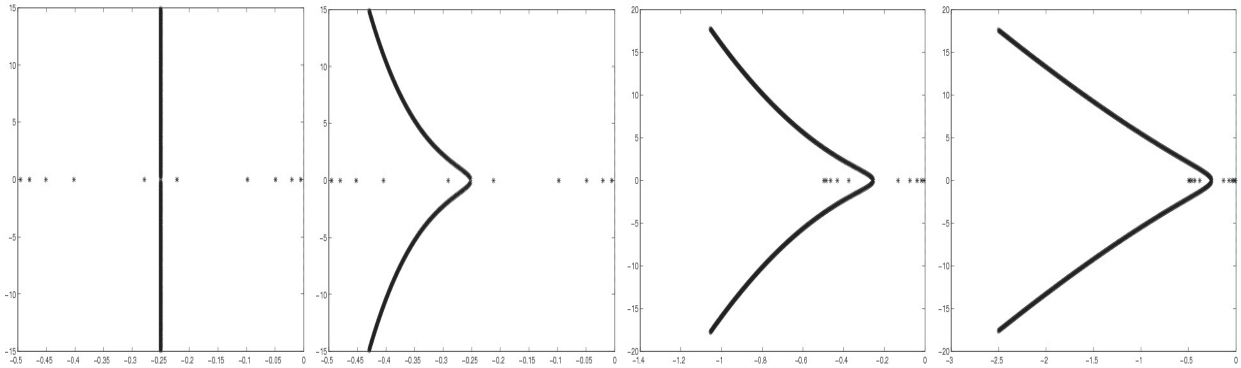

Fig. 2. From left to right: $\alpha = 0$ fluid, $\alpha = 0.1$, 0.25, 0.4 hyperbolic type.

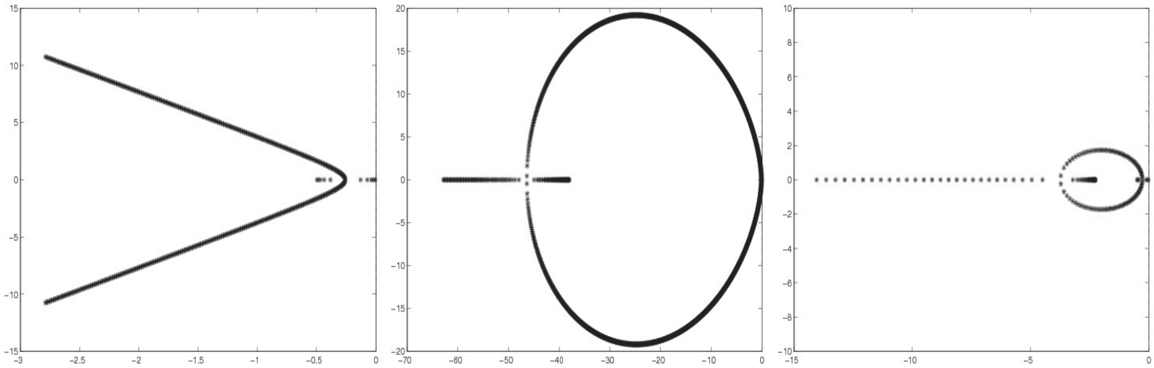

Fig. 3. From left to right: $\alpha = 0.5$ limiting case, $\alpha = 0.8$ diffusive type, and $\alpha = 1$ Rayleigh type.

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Appendix A. FRACTIONAL POWERS OF MATRICES

We recall the Spectral Theorem for symmetric real-valued matrices: if $A = A^T \in M_{n\times n}(\mathbb{R})$, then there exists a diagonal matrix Λ and an orthogonal matrix P, (i.e. $P^T P = I_n$), such that $A = P^{-1}\,\Lambda\,P$. Then, if $A = A^T \geq 0$, i.e. A is positive, then one can uniquely define

$$A^\alpha := P^{-1}\,\Lambda^\alpha\,P\,,$$

with $\Lambda^\alpha = \mathrm{diag}(\lambda_1^\alpha, ..., \lambda_n^\alpha)$, since $\lambda_i \geq 0$.