Puzzles in pipes with negative curvature: from the Webster PDE to stable numerical simulation in real time*

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Abstract

Minimal realizations of a class of delay-differential systems are derived for the digital simulation of waveguides, modelled by the Webster horn equation. Studying their stability is an interesting issue, since *negative* curvatures could lead to unstable systems. Spectral properties of Toeplitz matrix play a key role.

Keywords

Webster equation, delay-differential systems, BIBO-stability, simulation.

1 Introduction

The wave equation with space-varying coefficients that models propagation in horns is the Webster PDE, which is known to be *conservative*, whatever the shape of the horn; hence, stable numerical schemes can be derived for it, e.g. forward Euler on the vector (pressure, flow) with a CFL condition. But, in the case of *negative curvature*, the waveguide decompositions usually used for real-time simulations introduces unstable subsystems. A similar paradox in the case of a convergent cone has been recently fixed in [1] using minimal realization of a delay system. In the present work, the same methodology is applied to a pipe with negative (constant) curvature, the discretization of which now gives rise to a delay-differential system of retarded type.

2 1D propagation in a convex acoustic pipe

Consider a pipe with radius $r(\ell)=\cos(\ell)$ for $\ell \in [-\frac{L}{2}, \frac{L}{2}]$, $L < \pi$ (see Fig. 1a). Inside the pipe, the acoustic pressure p is governed by the (adimensionnal) Webster PDE

*This work is supported by the CONSONNES project, ANR-05-BLAN-0097-01
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$\{\partial_\ell^2 + 2\frac{r'(\ell)}{r(\ell)}\partial_\ell - \partial_t^2\}p(\ell,t) = 0$, the velocity v by the (adimensionnal) Euler PDE $\partial_t v(\ell,t) = -\partial_\ell p(\ell,t)$, and travelling waves $\phi^\pm := r(p \pm v)$ by the coupled equations

$$\partial_\ell \phi^\pm(\ell,t) \pm \partial_t \phi^\pm(\ell,t) = \big(r'(\ell)\,/\,r(\ell)\big)\,\phi^\mp(\ell,t). \tag{1}$$

The scattering matrix M such that $Y(s) = M(s)U(s)$ with $U(s) = [\hat{\phi}^+(\frac{-L}{2},s), \hat{\phi}^-(\frac{L}{2},s)]^T$ and $Y(s) = [\hat{\phi}^-(\frac{-L}{2},s), \hat{\phi}^+(\frac{L}{2},s)]^T$ in the Laplace domain is meromorphic in $\Gamma(s)^2 = s^2 - 1$, and though defining a BIBO stable system, as expected. But, the standard decomposition of M used for waveguide implementations involves functions of Γ with branching points 1 and –1. Hence, contrarily to flared pipes for which $\Gamma(s)^2 = s^2 + 1$, the analyticity is lost over $\mathbb{C}_0^+$: such a decomposition involves unstable sub-systems.

3 Approximation with pieces of conical pipes

Consider the piecewise affine approximations $\widetilde{r}_N$ of r with discretization step $\epsilon_N = \frac{L}{N}$. Waves in conical pieces of pipes are assumed to be ideally spherical and $\phi^\pm$ to be

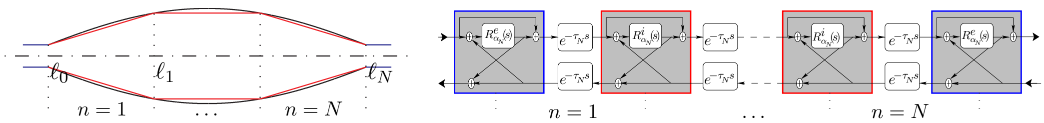

Figure 1: (a) Approximated shape of pipe, (b) Stable realization for simulation.

continuous at junctions. It leads to the network in Fig. 1b where reflection functions are $R_\alpha(s) = \frac{\alpha}{s-\alpha}$ with $\alpha_N^i = \frac{1-\cos\epsilon_N}{\epsilon_N} > 0$ for junctions inside the pipe and with $\alpha_N^e = \frac{\pi}{2\epsilon_N}(1 - \frac{\cos(L-\epsilon_N)}{\cos(L)}) < 0$ at both ends. This system corresponds to a delay differential system of retarded type with the following state-space representation

$$s\,X_N(s) = A_N\big(e^{-\tau_N s}\big)X_N(s) + B_N\big(e^{-\tau_N s}\big)U(s), \tag{2}$$
$$Y(s) = C_N\big(e^{-\tau_N s}\big)X_N(s) + D_N\big(e^{-\tau_N s}\big)U(s), \tag{3}$$

where X_N is defined from $\hat{\phi} = \hat{\phi}^+ + \hat{\phi}^-$ by $X_N(s) = \frac{\Delta_N}{s}[\hat{\phi}(\ell_0,s), \hat{\phi}(\ell_1,s), \ldots, \hat{\phi}(\ell_N,s)]^T$, and where $A_N(w) = \Delta_N W_N(w)$, $W_N(w)$ is the $(N+1)\times(N+1)$-symmetrical Toeplitz matrix such that $[W_N(w)]_{ij} = w^{|i-j|}$, and $\Delta_N = \mathrm{diag}(\big[\alpha_N^e, \alpha_N^i, \ldots, \alpha_N^i, \alpha_N^e\big])$.

For fixed N, the set $\mathcal{P}_N$ of roots of $\det\big(sI_{(N+1)} - A_N(e^{-\tau_N s})\big)$ is countable, belongs to $\mathbb{C}_0^-$ and the system in Fig. 1b proves to be stable. The theoretical question of the behaviour of the set of poles $\mathcal{P}_N$ as N goes towards infinity is left open, so far. Simulation results will be presented to illustrate and help understand their behaviour.

References

[1] R. Mignot, T. Hélie, D. Matignon. *Stable Realization of a Delay System Modeling a Convergent Acoustic Cone.* IEEE MED08. Ajaccio, France, 2008. p 1574-1579.

[2] E. Ducasse. *An alternative to the traveling-wave approach for use in two-port descriptions of acoustic bores.* J. Acoust. Soc. Am. 2002, Vol. 112. p 3031-3041.