On the singularities of fractional differential systems, using a mathematical limiting process based on physical grounds ‡

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Abstract.
Fractional systems are associated to irrational transfer functions for which non unique analytic continuations are available (from some right-half Laplace plane to a maximal domain). They involve continuous sets of singularities, namely cuts, which link fixed branching points with an arbitrary path. In this paper, an academic example of the 1D heat equation and a realistic model of an acoustic pipe on bounded domains are considered. Both involve a transfer function with a unique analytic continuation and singularities of pole type. The set of singularities degenerates into *uniquely* defined cuts, when the length of the physical domain becomes infinite. From a mathematical point of view, both the convergence in Hardy spaces of some right-half complex plane and the pointwise convergence are studied and proved.

Keywords: Boundary value problem, Generalized linear systems, Integral representations, Laplace transforms, Singularities, Heat equation, Scattering problems.

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1. Introduction

This paper is devoted to enlight links between some fractional differential systems and physical phenomena modelled by partial differential equations in unbounded domains. Two problems are considered: first, in § 2, the academic example of the 1D heat equation; second, in § 3, a realistic model of an acoustic pipe including visco-thermal losses at the walls and a varying cross-section with constant curvature.

On the one hand, the complex analysis of the transfer functions related to these problems reveals that singularities involve cuts between fixed branching points. On the other hand, the same problems considered on a bounded domain give rise to a countable set of poles. These are standard results.

The main point of this paper is to exhibit in which mathematical sense the cuts of the fractional systems under consideration can be viewed as the limit of a densification of the set of poles when the boundary of the domain goes towards infinity.

As a result, whereas the cuts of the transfer function of fractional systems can be chosen arbitrarily (starting from and ending at fixed branching points), the cuts defined from the limit of the countable set of poles are uniquely defined.

From a mathematical point of view, two types of convergence are examined: first, the convergence of the transfer functions in the Hardy space of some left half-complex plane is proved, second the pointwise convergence of some analytic continuation of the transfer functions is obtained over the whole Laplace plane except on these cuts.

2. A toy model: the heat equation

2.1. Physical model

Consider the following adimensional heat conduction problem on $I_\varepsilon = (0, 1/\varepsilon)$, described by,

$$\forall x \in I_\varepsilon,\, \forall t > 0, \quad \partial_t T(x,t) - \partial_x^2 T(x,t) = 0, \tag{1}$$

with null initial conditions,

$$\forall x \in I_\varepsilon, \quad T(x, t = 0) = 0, \tag{2}$$

a *controlled* Neumann boundary condition at $x = 0$,

$$\forall t > 0, \quad -\partial_x T(x = 0, t) = u(t), \tag{3}$$

a free Dirichlet boundary condition at $x = 1/\varepsilon$,

$$\forall t > 0, \quad T(x = 1/\varepsilon, t) = 0, \tag{4}$$

and a Dirichlet *observation* at $x = 0$ as output,

$$\forall t > 0, \quad y(t) := T(x = 0, t). \tag{5}$$

This models a finite bar, at rest before $t = 0$, controlled by a heat flux u and observed by the temperature at the left end $x = 0$, and for which the temperature is kept equal to zero at the right end $x = 1/\varepsilon$.

2.2. Transfer function and Hardy spaces

Following e.g. [1], this problem can be solved in the Laplace domain and yields the transfer function $H_\varepsilon(s) := \hat{y}(s)/\hat{u}(s)$ given by

$$\forall s \in \mathbb{C}_0^+, \quad H_\varepsilon(s) := \frac{\tanh(\sqrt{s}/\varepsilon)}{\sqrt{s}}, \tag{6}$$

where $\forall \alpha \in \mathbb{R}, \quad \mathbb{C}_\alpha^+ := \{s \in \mathbb{C} \,|\, \Re\mathrm{e}(s) > \alpha\}$, and where $\hat{T}$ and $\hat{u}$ respectively denote the one-sided Laplace transforms of T and u with respect to the time variable.

In (6), the square-root means the analytic continuation of the positive square-root on $\mathbb{R}^+$ on a domain compatible with the one-sided Laplace transform, namely $\mathbb{C}_0^+$.

$$\sqrt{\cdot}: \quad s = \rho \exp(i\theta) \mapsto \sqrt{\rho}\exp(i\theta/2) \tag{7}$$

with $(\rho, \theta) \in \mathbb{R}^{+*} \times (-\pi/2, \pi/2)$.

Following e.g. [2], let us introduce:

Definition 1. For $\alpha > 0$ and $m > 0$, $\mathbb{H}^m(\mathbb{C}_\alpha^+)$ denotes the Hardy space defined by

$$\mathbb{H}^m(\mathbb{C}_\alpha^+) = \Big\{ H : \mathbb{C}_\alpha^+ \to \mathbb{C} \,\Big|\, H \text{ is holomorphic in } \mathbb{C}_\alpha^+, \text{ and } \sup_{\zeta > \alpha} \int_{\mathbb{R}} |H(\zeta + i\omega)|^m \mathrm{d}\omega < \infty \Big\}. \tag{8}$$

The norm of $H \in \mathbb{H}^m(\mathbb{C}_\alpha^+)$ is then defined by

$$\|H\|_{\mathbb{H}^m(\mathbb{C}_\alpha^+)} := \sup_{\zeta > \alpha} \left[\frac{1}{2\pi} \int_{\mathbb{R}} |H(\zeta + i\omega)|^m \, \mathrm{d}\omega \right]^{\frac{1}{m}} . \tag{9}$$

Then, the following theorem holds.

Theorem 2. Let $\varepsilon > 0$, then

$$\forall \alpha \geq 0, \forall m > 2, \quad H_\varepsilon \in \mathbb{H}^m(\mathbb{C}_\alpha^+) . \tag{10}$$

Proof. Let $\varepsilon > 0$, $\alpha \geq 0$ and $m > 2$. Since

$$\forall s \in \mathbb{C}_\alpha^+, \quad H_\varepsilon(s) = \frac{1}{\sqrt{s}} \frac{1 - \exp(-2\sqrt{s}/\varepsilon)}{1 + \exp(-2\sqrt{s}/\varepsilon)}, \tag{11}$$

with (7), it follows that

$$|H_\varepsilon(s)|^m \sim |s|^{-\frac{m}{2}} \text{ as } |s| \to \infty, \tag{12}$$

$$|H_\varepsilon(s)|^m \sim \varepsilon^{-m} \quad \text{as } s \to 0. \tag{13}$$

Hence, $\int_{\zeta + i\mathbb{R}} |H_\varepsilon(s)|^m \, \mathrm{d}s$ is a finite integral for $m > 2$ (due to (12)) and $\alpha \geq 0$. □

Theorem 3. Function H_0 defined by

$$\begin{aligned} H_0 : \mathbb{C}_0^+ &\to \mathbb{C} \\ s &\mapsto 1/\sqrt{s} \end{aligned} \tag{14}$$

is analytic over $\mathbb{C}_0^+$.

$$\forall \alpha > 0, \forall m > 2, \quad H_0 \in \mathbb{H}^m(\mathbb{C}_\alpha^+) . \tag{15}$$

Moreover,

$$H_\varepsilon \overset{\mathbb{H}^m(\mathbb{C}_\alpha^+)}{\longrightarrow} H_0 \quad \text{as} \quad \varepsilon \to 0^+ . \tag{16}$$

Proof. Using (7), H_0 is well defined, and analytic in $\mathbb{C}_0^+$.
Proving (15) is analogous to proving (10), but contrarily to theorem 2, the case $\alpha = 0$ cannot be included here: condition $\alpha > 0$ ensures the convergence of the integral in (9) for $\omega \to 0$.
Now, as for (16), the behaviour of $\frac{1}{\sqrt{s}} \frac{2\,\exp(-2\sqrt{s}/\varepsilon)}{1+\exp(-2\sqrt{s}/\varepsilon)}$ has to be studied as $\varepsilon \to 0^+$. Denoting $z = \sqrt{s}/\varepsilon$, we get $|s^{-1/2}\,\frac{2e^{-2z}}{1+e^{-2z}}| \le 2|s|^{-1/2}\,e^{-2\,\Re e(z)}$, since $|1+e^{-2z}| \ge |1+\Re e(e^{-2z})| \ge 1$. Then, for $s = \zeta + i\omega$ and $\zeta \ge \alpha$, we have $e^{-2\,\Re e(z)} \le e^{-2\cos(\pi/4)\,\sqrt{\alpha}/\varepsilon}$. Raising the latter bound to power m and using (15) yields the desired result. □

2.3. Complex analysis and analytic continuations

The transformation $\sqrt{s} \mapsto -\sqrt{s}$ keeps H_ε invariant, so that H_ε is a function of s only. More precisely, H_ε can be analytically continued on the domain $\mathcal{D}_\varepsilon$ given by

$$\mathcal{D}_\varepsilon = \mathbb{C} \setminus \mathcal{P}_\varepsilon, \tag{17}$$

$$\mathcal{P}_\varepsilon = \{s_n = -\varepsilon^2\,(n+\frac{1}{2})^2\,\pi^2 \,\big|\, n \in \mathbb{N}\}, \tag{18}$$

where $\mathcal{P}_\varepsilon$ is the countable set of poles of H_ε. Note that $0 \notin \mathcal{P}_\varepsilon$ and $H_\varepsilon(0) = 1$. The set of zeros of H_ε is given by

$$\mathcal{Z}_\varepsilon = \{\zeta_n = -\varepsilon^2\,n^2\,\pi^2 \,\big|\, n \in \mathbb{N}^*\}. \tag{19}$$

Using formula $\tanh(z)/z = \sum_{n\in\mathbb{N}} \left(z^2 + (n+\frac{1}{2})^2\,\pi^2\right)^{-1}$ from e.g. [3], H_ε proves to be a meromorphic function which can be expanded into

$$\begin{array}{llll} H_\varepsilon : \mathbb{C}\setminus\mathcal{P}_\varepsilon \to \mathbb{C} \\ \qquad s \qquad\quad \mapsto \displaystyle\sum_{n\in\mathbb{N}} \frac{\varepsilon}{s + \varepsilon^2(n+\frac{1}{2})^2\pi^2} \end{array} \tag{20}$$

Note the difference between (6) and (20): the latter is the unique maximal analytic continuation of the former.
Remark: Poles $\mathcal{P}_\varepsilon$ and zeroes $\mathcal{Z}_\varepsilon$ are *intertwined* on the negative real axis $\mathbb{R}^-$, as already noticed in e.g. [4]. Moreover, in a mathematically rigorous way,

$$\overline{\cup_{\varepsilon>0}\mathcal{P}_\varepsilon} = \mathbb{R}^-. \tag{21}$$

This is the reason why we now define $\mathcal{D}_0 := \mathbb{C}\setminus\mathbb{R}^-$ and

$$\begin{array}{ll} H_0 : \mathbb{C}\setminus\mathbb{R}^- \to \mathbb{C} \\ \qquad s \qquad\quad \mapsto \displaystyle\int_0^\infty \frac{1}{\pi\sqrt{\xi}}\,\frac{1}{s+\xi}\,\mathrm{d}\xi \end{array} \tag{22}$$

Using the links between fractional calculus and *diffusive representations*, as in [5], it can be proved‖ that $H_0(s)$ as defined by (22) also has the value $s^{-1/2}$, as expected, once definition (7) has been *analytically continued* to $(\rho,\theta) \in \mathbb{R}^{+*} \times (-\pi,\pi)$.

Note the difference between (14) and (22): the latter is *a* maximal analytic continuation of the former, but it is certainly not unique! It is well known that *any* branch cut between the branchpoints $s = 0$ and another branchpoint at infinity in $\Re e(s) < 0$ would also do (see e.g. [5, 6]). Among these analytic continuations, (22) defines the unique limit of H_ε on $\mathcal{D}_\varepsilon$, for $\varepsilon \to 0^+$, as stated in the following theorem.

‖ an elementary proof goes as follows: substitute $x = \sqrt{\xi/s}$ in the numerical identity $\int_0^\infty \left(1+x^2\right)^{-1} \mathrm{d}x = \pi/2$ for any $s \in \mathbb{R}^{+*}$, and get (22); then perform an analytic continuation from $\mathbb{R}^{+*}$ to $\mathbb{C}\setminus\mathbb{R}^-$ for both sides of the identity in the variable s.

Theorem 4. Let $\varepsilon > 0$, then $\mathcal{D}_0 \subset \mathcal{D}_\varepsilon$, and

$$\forall s \in \mathbb{C} \setminus \mathbb{R}^-, \quad \lim_{\varepsilon \to 0^+} H_\varepsilon(s) = H_0(s). \tag{23}$$

Proof. Let s be fixed in $\mathbb{C} \setminus \mathbb{R}^-$. Then, $\forall \varepsilon > 0$, H_ε is analytic at s, and from the *extended* definition of (7) above, $\Re e(\sqrt{s}) > 0$, so that $\lim_{\varepsilon \to 0^+} \exp(-2\sqrt{s}/\varepsilon) = 0$. The limit of (11) for $\varepsilon \to 0^+$ yields the desired result. □

Remark: The cut $\mathcal{C}$, which appears as the *limit set* of singularities of physical problems on bounded domains, is characterized by

$$\Re e(\sqrt{s}) = 0 \Leftrightarrow s \in \mathcal{C} := \mathbb{R}^-. \tag{24}$$

2.4. Integral representations and interpretations

Well-posed integral representations of both H_ε and H_0 are given by

$$H_\varepsilon(s) = \int_0^\infty \frac{1}{s+\xi}\, \mathrm{d}\mu_\varepsilon(\xi). \tag{25}$$

From e.g. [7], the well-posedness condition reads $\int_0^\infty \frac{1}{1+\xi}\, \mathrm{d}\mu(\xi) < \infty$; it is fulfilled by the associated measures μ_ε and μ_0, defined as follows:

$$\mu_\varepsilon \quad = \sum_{n \in \mathbb{N}} 2\varepsilon\, \delta_{-s_n}(\xi), \quad \text{for } \varepsilon > 0\,, \tag{26}$$

$$\mathrm{d}\mu_0(\xi) = \frac{1}{\pi\sqrt{\xi}}\, \mathrm{d}\xi\,. \tag{27}$$

For $\varepsilon > 0$, μ_ε is a *discretely supported* measure at points $\xi_n = -s_n = \varepsilon^2(n+\frac{1}{2})^2\pi^2$, $n \in \mathbb{N}$; whereas μ_0 is *absolutely continuous* w.r.t Lebesgue measure on $\mathbb{R}^+$. Both these systems are presented in examples 2.1 and 2.2 of [8], and fully analyzed as well-posed systems in examples 3.2 and 3.4 of [7].

We have the following convergence theorem for the associated measures:

Theorem 5. The *weak* convergence of measures holds:

$$\mu_\varepsilon \xrightarrow{\mathrm{w}} \mu_0, \quad \text{as} \quad \varepsilon \to 0^+. \tag{28}$$

Hence, for $s \in \mathcal{D}_0$ and with $\varphi_s(\xi) := \frac{1}{s+\xi}$ as test function in $\mathcal{C}_0(\mathbb{R}^+)$, we recover $H_\varepsilon(s) \longrightarrow H_0(s)$, as $\varepsilon \to 0^+$.

Proof. Let $\varphi \in \mathcal{C}_\mathrm{c}(\mathbb{R}^+_\xi)$, we compute $< \mu_0, \varphi >= \int_0^\infty \frac{1}{\pi\sqrt{\xi}}\, \varphi(\xi)\, \mathrm{d}\xi$ on the one hand, and $< \mu_\varepsilon, \varphi >= \sum_{n=0}^\infty 2\varepsilon\, \varphi(\xi_n)$ on the other hand. With the change of variables $\xi = x^2\, \pi^2$, the test function reads $\psi(x) := \varphi(\xi)$ and still belongs to $\mathcal{C}_\mathrm{c}(\mathbb{R}^+_x)$. The only thing to prove is then

$$2\varepsilon \sum_{n=0}^\infty \psi(\varepsilon\, (n+\frac{1}{2})) \longrightarrow 2 \int_0^\infty \psi(x)\, \mathrm{d}x\,,$$

as $\varepsilon \to 0^+$, which is nothing but the limit of a Riemann sum. One can also try to extend the previous result to $\varphi \in \mathcal{C}_0(\mathbb{R}^+_\xi)$, and not only $\mathcal{C}_\mathrm{c}(\mathbb{R}^+_\xi)$.
Nevertheless, the well-posedness conditions help prove the last item, even if $\varphi_s \notin \mathcal{C}_\mathrm{c}(\mathbb{R}^+_\xi)$, but $\varphi_s \in \mathcal{C}_0(\mathbb{R}^+_\xi)$. □

3. A more involved model in acoustics

3.1. Acoustic model of a piece of pipe

Acoustic model Consider a mono-dimensional model of linear acoustic propagation in axisymmetric pipes, which takes into account the visco-thermal losses and the varying cross section. The acoustic pressure p and velocity v are governed by the *Webster-Lokshin* equation (see [9, 10] referred to in [11], and more recently [12] for the Webster component) and Euler equation, given by, in the Laplace domain,

$$\left[\left(\left(\frac{s}{c_0}\right)^2 + 2\eta(\ell)\left(\frac{s}{c_0}\right)^{\frac{3}{2}} + \Upsilon(\ell)\right) - \partial_\ell^2\right]\big(r(\ell)p(\ell,s)\big) = 0, \tag{29}$$

$$\rho_0\, s\, v(\ell,s) + \partial_\ell p(\ell,s) = 0, \tag{30}$$

where s is the Laplace variable, ℓ is the *curvilinear* abscissa of the wall, c_0 is the speed of sound, ρ_0 is the mass density, $r(\ell)$ is the radius of the pipe. η quantifies the effect of the visco-thermal losses, and $\Upsilon = r''/r$ is the curvature of the horn.

Note that the symbol $s^{3/2}$ is the Laplace transform of the fractional time derivative $\partial_t^{3/2}$, as introduced in e.g. [13]. The role of the parameter η alone, when $\Upsilon = 0$ has been fully understood in [14], in which three closed-form solutions of this problem have been obtained. The diffusive phenomenon in which we are interested in this section, is actually due to the curvature $\Upsilon(\ell)$ and requires special treatment.

Let $\psi^+(\ell,s)$ and $\psi^-(\ell,s)$ be defined by

$$\begin{bmatrix} \psi^+ \\ \psi^- \end{bmatrix} = \frac{r}{2}\begin{bmatrix} 1 & \rho_0 c_0 \\ 1 & -\rho_0 c_0 \end{bmatrix}\begin{bmatrix} p \\ v \end{bmatrix} + \left[\frac{c_0 r'}{2rs}\right]\begin{bmatrix} -1 \\ 1 \end{bmatrix} p. \tag{31}$$

This alternative acoustic state defines travelling waves ψ^+ and ψ^- which extend the usual decoupled ingoing and outgoing planar or spherical waves propagating in straight or conical lossless pipes respectively ($\eta = 0, \Upsilon = 0$).

Adimensional problem for a piece of pipe Consider a section of horn with length L, constant positive curvature $\Upsilon > 0$, and constant losses coefficient η. Let us define the adimensional variables and coefficients for this horn:

$$\underline{\boldsymbol{\ell}} = \frac{\ell}{L}, \quad \underline{\boldsymbol{s}} = \frac{s}{c_0\sqrt{\Upsilon}}, \quad \varepsilon = \frac{1}{L\sqrt{\Upsilon}}, \quad \text{and } \beta = \frac{\eta}{\sqrt[4]{\Upsilon}},$$

and for any dimensional function F, let us define its adimensional version $\underline{F}$ such as $\underline{\boldsymbol{F}}(\underline{\boldsymbol{\ell}},\underline{\boldsymbol{s}}) = F(\ell,s)$.

Υ becomes $\underline{\boldsymbol{\Upsilon}} = 1$ and equations (29), (30) and (31) become, for all $\underline{\boldsymbol{\ell}} \in (0,1)$,

$$\left[\left(\underline{\boldsymbol{s}}^2 + 2\,\beta\underline{\boldsymbol{s}}^{\frac{3}{2}} + 1\right) - \varepsilon^2\partial_{\underline{\boldsymbol{\ell}}}^2\right]\left(\underline{\boldsymbol{r}}\,\underline{\boldsymbol{p}}\right) = 0, \tag{32}$$

$$\rho_0 c_0\, \underline{\boldsymbol{s}}\,\underline{\boldsymbol{v}} + \varepsilon\partial_{\underline{\boldsymbol{\ell}}}\underline{\boldsymbol{p}} = 0, \tag{33}$$

$$\begin{bmatrix} \underline{\boldsymbol{\psi}}^+ \\ \underline{\boldsymbol{\psi}}^- \end{bmatrix} = \frac{\underline{\boldsymbol{r}}}{2}\begin{bmatrix} 1 & \rho_0 c_0 \\ 1 & -\rho_0 c_0 \end{bmatrix}\begin{bmatrix} \underline{\boldsymbol{p}} \\ \underline{\boldsymbol{v}} \end{bmatrix} + \frac{c_0\underline{\boldsymbol{r}}'}{2\underline{\boldsymbol{r}}\,\underline{\boldsymbol{s}}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\underline{\boldsymbol{p}}. \tag{34}$$

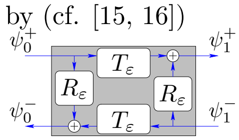
Note that $\varepsilon = 1/(L\sqrt{\Upsilon})$ is conversely proportional to L: it will play the same role as ε in section 2 to compute some limit transfer function when $\varepsilon \to 0^+$.

In the following we only consider adimensional problems, the notation $\underline{\boldsymbol{X}}$ is re-noted X for sake of legibility.

3.2. Transfer functions and Hardy spaces

3.2.1. Two-port representation and reflection function Solving (32)-(34) for $\ell \in (0,1)$ with zero initial conditions, controlled boundary conditions $\psi_0^+(s) := \psi^+(\ell = 0, s)$ (incoming wave at $\ell = 0$) and $\psi_1^-(s) := \psi^-(\ell = 1, s)$ (incoming wave at $\ell = 1$), and observing the outgoing travelling waves (ψ_1^+, ψ_0^-), lead to the solution $[\psi_1^+, \psi_0^-]^T = \mathbf{Q}_\varepsilon.[\psi_0^+, \psi_1^-]^T$. The *scattering matrix* $\mathbf{Q}_\varepsilon$ is given by (cf. [15, 16])

$$\mathbf{Q}_\varepsilon(s) = \begin{bmatrix} T_\varepsilon(s) & R_\varepsilon(s) \\ R_\varepsilon(s) & T_\varepsilon(s) \end{bmatrix}.$$



Both T_ε and R_ε are intricate transfer functions which are not convenient to use for simulation purposes in time domain. Moreover, everything depends on ε, which mixes the effects and make things difficult to analyze.

In the following we are only interested in the transfer function $R_\varepsilon(s) = \psi_0^-(s)/\psi_0^+(s)$, which represents the global reflection on the travelling waves of the horn at the left end (ℓ=0).

The function R_ε is given by, $\forall s \in \mathbb{C}_0^+$,

$$R_\varepsilon(s) = \frac{\frac{1}{2}\left(\frac{s}{\Gamma(s)} - \frac{\Gamma(s)}{s}\right)\sinh\left(\frac{\Gamma(s)}{\varepsilon}\right)}{\cosh\left(\frac{\Gamma(s)}{\varepsilon}\right) + \frac{1}{2}\left(\frac{s}{\Gamma(s)} + \frac{\Gamma(s)}{s}\right)\sinh\left(\frac{\Gamma(s)}{\varepsilon}\right)}, \tag{35}$$

$$= \frac{\frac{1}{2}\left(\frac{s}{\Gamma(s)} - \frac{\Gamma(s)}{s}\right)\tanh\left(\frac{\Gamma(s)}{\varepsilon}\right)}{1 + \frac{1}{2}\left(\frac{s}{\Gamma(s)} + \frac{\Gamma(s)}{s}\right)\tanh\left(\frac{\Gamma(s)}{\varepsilon}\right)}, \tag{36}$$

where, as for equation (7) of section 2, in (35), $\Gamma(s)$ denotes the analytic continuation of the positive square-root of

$$\Gamma(s)^2 = s^2 + 2\beta s^{\frac{3}{2}} + 1, \tag{37}$$

on the domain $\mathbb{C}_0^+$ which is compatible with the one-sided Laplace transform.
Remark: Now, in the part of the complex plane for which $\Re e(\Gamma(s)) > 0$, letting $z := \Gamma(s)/\varepsilon$, working on formula (36), we find as in section 2, that, as $\varepsilon \to 0^+$,

$$R_\varepsilon(s) \to \frac{\frac{1}{2}\left(\frac{s}{\Gamma(s)} - \frac{\Gamma(s)}{s}\right)}{1 + \frac{1}{2}\left(\frac{s}{\Gamma(s)} + \frac{\Gamma(s)}{s}\right)} = \frac{s^2 - \Gamma(s)^2}{2s\Gamma(s) + s^2 + \Gamma(s)^2}$$

$$= \frac{s - \Gamma(s)}{s + \Gamma(s)} := R_0(s)\,. \tag{38}$$

3.2.2. Physical interpretation To reduce the simulation cost, a decomposition of the two-port $\mathbf{Q}_\varepsilon$ into elementary transfer functions can be looked for. Using the method presented in [17], we get the framework of figure 1.

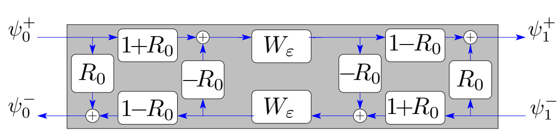


Figure 1. Decomposition of the two-port $\mathbf{Q}_\varepsilon$

Here, $R_0(s)$, already defined by (38), represents the wave reflection at the interfaces of the horn and

$$W_\varepsilon(s) := e^{-\Gamma(s)/\varepsilon} \tag{39}$$

represents the propagation through the horn. For the present work, this framework is of interest because the parameter ε is now clearly *isolated* in W_ε only. We obtain another algebraic expression for R_ε, namely:

$$R_\varepsilon = R_0 \frac{1 - W_\varepsilon^2}{1 - {R_0}^2\, W_\varepsilon^2}, \tag{40}$$

which helps prove the pointwise convergence result below. Note that, whereas the functions R_0 and W_ε of the decomposition depend on $\Gamma(s)$, R_ε is a function of $\Gamma(s)^2$, and s only, see (36).

Since $L = 1/(\varepsilon\sqrt{\Upsilon})$, $L \to \infty$ as $\varepsilon \to 0$. The pointwise convergence allows us to interpret function R_0 as the waves reflection of a *semi-infinite* horn, which is anechoic ($W_\varepsilon(s) \to 0$ as $\varepsilon \to 0^+$, as soon as $\Re e(\Gamma(s)) > 0$).

We are now in a position to perform a detailed complex analysis of both R_ε and R_0 in terms of poles, zeroes, branching points and cuts, and analyze their evolution with respect to ε.

Hardy spaces Now, we give some properties of the transfer functions in $\mathbb{C}_0^+$. In [16], the following results have been proved for $\varepsilon > 0$:

$$\forall s \in \mathbb{C}_0^+, \qquad \Re e(\Gamma(s) - s) > 0, \tag{41}$$

$$\forall s \in \mathbb{C}_0^+, \qquad |R_0(s)| < 1 \text{ and } |W_\varepsilon(s)| < 1, \tag{42}$$

$$\forall m > 0, \qquad W_\varepsilon \in \mathbb{H}^m(\mathbb{C}_0^+)\,, \tag{43}$$

$$\forall m > 2, \qquad R_0 \in \mathbb{H}^m(\mathbb{C}_0^+) \text{ and } R_\varepsilon \in \mathbb{H}^m(\mathbb{C}_0^+)\,. \tag{44}$$

Moreover, the following result holds:

Theorem 6. Let $\varepsilon > 0$,

$$\forall \alpha > 0, \forall m > 2, \quad R_\varepsilon \overset{\mathbb{H}^m(\mathbb{C}_\alpha^+)}{\longrightarrow} R_0 \quad \text{as} \quad \varepsilon \to 0^+\,. \tag{45}$$

Proof. Let $s \in \mathbb{C}_\alpha^+$. From (41), $|W_\varepsilon(s)| < e^{-\Re e(s)/\varepsilon} < e^{-\alpha/\varepsilon}$, then with (42), we get $|\frac{1-R_0^2}{1-R_0^2 W_\varepsilon^2}| < \frac{1}{1-e^{-2\alpha/\varepsilon}}$. Consequently $|R_0 - R_\varepsilon| = |R_0 \frac{1-R_0^2}{1-R_0^2 W_\varepsilon^2} W_\varepsilon^2 \;| < |R_0| \frac{e^{-2\alpha/\varepsilon}}{1-e^{-2\alpha/\varepsilon}}$. Now, from (44), $R_0 \in \mathbb{H}^m(\mathbb{C}_0^+) \subset \mathbb{H}^m(\mathbb{C}_\alpha^+)$. Finally, $\|R_0 - R_\varepsilon\|_{\mathbb{H}^m(\mathbb{C}_\alpha^+)} < \|R_0\|_{\mathbb{H}^m(\mathbb{C}_\alpha^+)} \frac{e^{-2\alpha/\varepsilon}}{1-e^{-2\alpha/\varepsilon}} \to 0$, when $\varepsilon \to 0^+$ for $m > 2$. $\square$

3.3. Branching points and Cuts of $\Gamma(s)$

In this subsection, we discuss the possible analytic continuations of function Γ in the whole Laplace domain.

3.3.1. Cut on $\mathbb{R}^-$ From (37) and because of $s^{3/2}$, Γ^2 has a cut which links $s=0$ and $s=\infty$ in $\mathbb{C}_0^-$. As it has been done in section 2, we choose the cut on $\mathbb{R}^-$. Note that this choice is required to ensure the hermitian symmetry, causality and stability of the transfer functions.

3.3.2. Symmetric cut Function $\Gamma(s)$ has some other branching points, which are solutions of $\Gamma(s)^2 = 0$. In the appendix of [15], it has been checked that there are 2 conjugate solutions with negative real part. Now we must choose a cut which links these branching points.

To ensure hermitian symmetry, causality and stability of transfer functions, the cut must satisfy two constraints:

(C1) the cut must be symmetric w.r.t $\mathbb{R}$,

(C2) the cut must lie in $\mathbb{C}_0^-$ only.

Figure 2 below shows two different such choices.

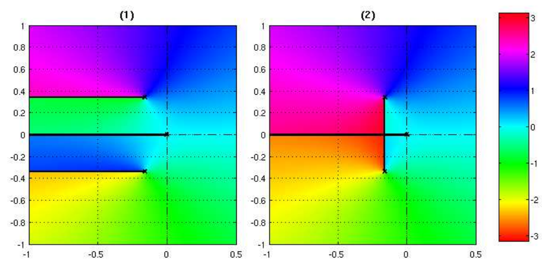


Figure 2. Phase of $\Gamma(s)$ in the complex plane ($\beta = 2$). (1) horizontal cut, (2) vertical cut.

Positive square root In the sequel, we will make use of a special choice of the symmetric cut, given by:

$$\Gamma(s) := \sqrt{s^2 + 2\beta s^{\frac{3}{2}} + 1}, \tag{46}$$

where $\sqrt{.}$ stands for *the* holomorphic extension to $(\rho, \theta) \in \mathbb{R}^{+*} \times (-\pi, \pi)$ of the square root defined by (7).

Defining $\Gamma(s)$ by (46), function Γ is holomorphic in $\mathbb{C} \setminus (\mathbb{R}^- \cup \mathcal{C})$, with $\mathcal{C} := \{s \in \mathbb{C} / \Gamma(s)^2 \in \mathbb{R}^-\}$. The cuts are $\mathbb{R}^-$ and $\mathcal{C}$ (cf. figure 3). Note that $\mathcal{C}$ has been proved to be included in $\mathbb{C}_0^-$, and its geometry only depends on the coefficient $\beta > 0$ (for the adimensional problem).

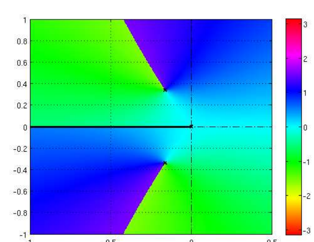


Figure 3. Phase of $\Gamma(s)$, as defined by (46), ($\beta = 2$).

The important property of definition (46) of Γ, is that

$$\Re e(\Gamma(s)) > 0, \ \forall s \in \mathbb{C} \setminus \mathcal{C}. \tag{47}$$

3.4. Poles, zeroes and convergence

Poles and Zeroes of R_ε Recall that the transformation $\Gamma \mapsto -\Gamma$ keeps R_ε invariant, so that R_ε is a function of Γ^2 only; more precisely, R_ε can be analytically continued on the domain $\mathcal{D}_\varepsilon$ given by

$$\mathcal{D}_\varepsilon = \mathbb{C} \setminus \left(\mathbb{R}^- \cup \mathcal{P}_\varepsilon\right), \tag{48}$$

$$\mathcal{P}_\varepsilon = \left\{ s \in \mathbb{C} \ / \ \frac{\tanh(\Gamma(s)/\varepsilon)}{\Gamma(s)} = \frac{-2s}{s^2 + \Gamma(s)^2} \right\}. \tag{49}$$

$\mathcal{P}_\varepsilon$ is a set singularities of R_ε. Unfortunately, it is difficult to study them explicitly, however numerical simulation makes the following conjecture plausible

Conjecture: Elements of $\mathcal{P}_\varepsilon$ are isolated singularities, and there are infinitely many such poles. As a corollary, there is no accumulation point. Let $\mathcal{P}_\varepsilon$ now denotes the set of *poles* of R_ε.

From (35), the set of zeroes of R_ε is $\mathcal{Z}_\varepsilon \cup \{\zeta_0, \overline{\zeta_0}\}$, where

$$\mathcal{Z}_\varepsilon = \left\{ \zeta_n \text{ and } \overline{\zeta_n} \in \mathbb{C} \ / \ \Gamma(\zeta_n)^2 = -\varepsilon^2 n^2 \pi^2 \mid n \in \mathbb{N}^* \right\},$$

$$\zeta_0 = (2\beta)^{-3/2} e^{2i\pi/3}. \tag{50}$$

ζ_0 is solution of $\Gamma(s) + s = 0$, when Γ is defined by (46), and we notice that elements of $\mathcal{Z}_\varepsilon$ lie on $\mathcal{C}$ (ie. $\mathcal{Z}_\varepsilon \subset \mathcal{C}$).

Pointwise convergence As already discussed in section 3.3 (§2), the analytic continuation of Γ is not unique, and so for R_0. Nevertheless, similarly to section 2, R_0 with Γ defined by (46), corresponds to the unique limit of R_ε defined in $\mathcal{D}_\varepsilon$, for $\varepsilon \to 0^+$, as stated in the following theorem

Theorem 7. Let the open set $\mathcal{D}_0$ and the analytic function R_0 be defined by

$$\mathcal{D}_0 := \mathbb{C} \setminus \left(\mathbb{R}^- \cup \mathcal{C} \cup \{\zeta_0, \overline{\zeta_0}\}\right) \tag{51}$$

$$\begin{aligned} R_0 : \mathcal{D}_0 &\to \mathbb{C} \\ s &\mapsto \frac{s - \Gamma(s)}{s + \Gamma(s)} \end{aligned} \tag{52}$$

with the function Γ, as defined by (46).

Then, $\forall \varepsilon > 0$,

$$\forall s \in \mathcal{D}_0, \lim_{\varepsilon \to 0^+} R_\varepsilon(s) = R_0(s). \tag{53}$$

Proof. Let $s \in \mathcal{D}_0$, and $\Gamma(s)$ defined by (46); from (47), $\Re e(\Gamma(s)) > 0$, so that $e^{-2\Gamma(s)/\varepsilon} \to 0$ when $\varepsilon \to 0^+$, as first noted in section 3.2 (§3). □

Note that ζ_0 and $\overline{\zeta_0}$, two zeroes of R_ε, are actually the two unique poles of R_0.

Remark: The cut $\mathcal{C}$, which appears as the *limit set* of singularities of physical problems on bounded domains, is characterized by

$$\Re e(\Gamma(s)) = 0 \Leftrightarrow s \in \mathcal{C}. \tag{54}$$

Unfortunately, we have not succeeded to prove the convergence of poles of R_ε to the cut $\mathcal{C}$ and $\{\zeta_0, \overline{\zeta_0}\}$ (poles of R_0). However numerical simulations illustrate this phenomenon: see figure 4 where poles and zeroes are represented by white and black dots respectively.

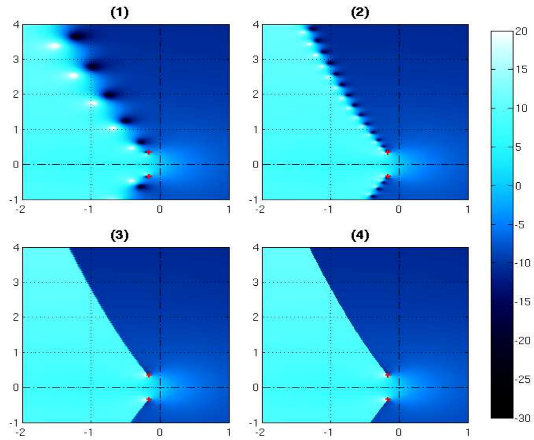


Figure 4. Modulus $|R_\varepsilon(s)|$ in the complex plane ($\beta = 2$). (1) $\varepsilon = 0.4$, (2) $\varepsilon = 0.15$, (3) $\varepsilon = 0.03$, (4) $\varepsilon = 0$.

Proposition 8. The poles of R_ε move continuously towards the cut $\mathcal{C}$ when parameter ε varies continuously.
Proof. Using the analytical continuation of (40) defined by (46), we get an equivalent definition of P_ε which is $\{s \in \mathbb{C} \;/\; R_0(s)^2 W_\varepsilon(s)^2 = 1\}$. Since $W_\varepsilon(s) \to 0$ when $\varepsilon \to 0^+$ $\forall s \in \mathbb{C} \setminus \mathcal{C}$, the poles of R_ε move to $\mathcal{C}$ or to the solutions of $|R_0(s)| = \infty$ (ζ_0 and $\overline{\zeta_0}$).

4. Conclusion

Some irrational transfer functions H_0 have been derived as the limit of solutions H_ε of physical boundary problems, on domains $(0, 1/\varepsilon)$. The convergence of H_ε towards H_0 in Hardy spaces for some Laplace right half-plane has been proved.

On the one hand, the limit transfer functions H_0 are those of causal fractional systems, for which infinitely many analytic continuations are available. Indeed, the set of singularities involve cuts which can be arbitrarily chosen between fixed branching points, in $\Re\mathrm{e}(s) \leq 0$.

On the other hand, the maximal analytic continuations of H_ε and their singularities of pole type are unique. As a main result of this paper (theoretically for model 1 and numerically for model 2), their pointwise limit uniquely defines a particular maximal analytic continuation of H_0 for which the cuts are hermitian symmetrical and described by a characteristic equation (see (24) and (54)). Moreover, integral representations of H_ε and of H_0 are available, and their corresponding measures are such that μ_ε converges towards μ_0, in a *weak* sense.

However, open questions arise from this work. First, we still have to prove the conjecture (numerically observed) stating that, for $\varepsilon > 0$, the singularities of the acoustic model define an infinite countable set of isolated poles. Second, estimating these poles and their residues should be studied to define discrete measures μ_ε and analyze their weak convergence towards a limit measure. A technical difficulty is that, contrarily to the heat conduction example, the poles approach the cut defined from the limit set but they do *not* belong to this cut. Hence, the test functions will have to be carefully chosen. Third, for both examples, we observe that the cuts correspond to a limit set of poles but also to a limit set of zeros (which are intertwined with poles for example 1). This matches with widely-used approximations of fractional operators which use placement of intertwined poles and zeros, as in e.g. [4]. A question is then: is this property generally satisfied, or are there some cases for which cuts correspond to limit sets of singularities exclusively (but not zeros) ? Fourth, the unique limit sets of singularities are obtained from the sequence of physically meaningful problems. Questions are then: can distinct sequences of physically meaningful causal problems lead to the same transfer function in $\mathbb{C}_0^+$ but different limit sets of singularities in $\mathbb{C}_0^-$? If not, how can this limit set be characterized ? It should be noted that once a state-space representation $\dot{X} = \mathcal{A}_\varepsilon X$ has been chosen for a sequence of physically meaningful PDE problems, then all the singularities of any transfer functions built from a system with a control operator $\mathcal{B}_\varepsilon$ and an observation operator $\mathcal{C}_\varepsilon$ do belong to $\mathrm{spec}(\mathcal{A}_\varepsilon)$: only point spectrum for $\varepsilon > 0$ and continuous spectrum for $\varepsilon = 0$. Hence, $\mathrm{spec}(\mathcal{A}_\varepsilon)$ fixes the general location of singularities, see *e.g.* [6]: this last remark should help obtain relevant information for our questions.

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