19th INTERNATIONAL CONGRESS ON ACOUSTICS
MADRID, 2–7 SEPTEMBER 2007

SOUND SYNTHESIS OF A NONLINEAR STRING USING VOLTERRA SERIES

PACS 43.40.Ga

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ABSTRACT

This paper proposes to solve and simulate various Kirchhoff models of nonlinear strings thanks to Volterra series. Two nonlinearities are studied : the string tension is supposed to depend either on the global elongation of the string, either on the local strain located at x. For each model, a Volterra series is used to represent the displacement as a functional of excitation forces. The Volterra kernels are solved in the Laplace domain thanks to a modal decomposition. As a last step, systematic identifications on these kernels lead to a structure in the time domain, which is composed of "linear filters", instantaneous sums and instantaneous products. Such a structure allows for sound synthesis thanks to standard signal processing techniques. The nonlinear dynamics introduced by this simulation is significant and perceptible on sounds for sufficiently large excitations.

INTRODUCTION

In musical acoustics, sound synthesis aims to reach more and more realism for complex systems such as musical instruments. Usually, these instruments involve nonlinear propagation phenomena as soon as vibrations are sufficiently large in *e.g.* gongs, dynamics of bowed strings, piano soundboards, etc. Thus, physical models which include nonlinear phenomena have been derived and usually solved using numerical methods, *e.g.* in [1].

During the 19th century, Kirchhoff derived a model of a one-dimensional perfectly flexible string including a nonlinearity due to the variation of tension [7]. This model has been re-investigated by Carrier in his paper [3] from which new string models of musical instruments have been elaborated. A. Watzky in [11] unified many works about nonlinear models with a three-dimensional model of a nonlinear stiff string. This generalization includes a torsion coupling and allows to introduce inharmonicity under the hypothesis of a linear elastic behavior. Advanced models and experimental results about strings can also be found in [10].

This article introduces the Volterra series to solve nonlinear models of damped strings. The Volterra series allow to represent physical models and perform analytical and numerical solutions. The kernels are used to compute the solution at a chosen order, with a structure of linear filters, instantaneous sums and products.

After the presentation of the various physical models of strings, an introduction to Volterra series will be made. The third section will show the complete procedure to calculate Volterra kernels for the first model. Results will be given for the two others. Finally, a comment about sound synthesis will be made before conclusion and perspectives.

PHYSICAL MODELS

Three dimensionless models will be studied with two kinds of nonlinearities (NL1,NL2) and two kinds of boundary conditions (B1,B2). Kirchhoff equation with an integro-differential term defines a "global" nonlinearity (NL1) whereas in the third model there is no integral, thus defining a "local" nonlinearity (NL2).

– The first model (NL1,B1) (cf. [5]) is the Kirchhoff equation including structural and fluid damping forces. The string is clamped at its extremities (B1) and motionless for $t \leq 0$. The dimensionless model is given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \beta \frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} = \left[1 + \epsilon \int_0^1 \|\frac{\partial u}{\partial x}\|^2 \mathrm{d}x\right] \frac{\partial^2 u}{\partial x^2} + \phi(x) f(t), \quad \forall (x,t) \in \Omega =]0;1[\times \mathbb{R}^{+*} \\ u(x,t) = 0 \; \forall (x,t) \in \{0;1\} \times \mathbb{R}^+, \\ \partial_t^k u(x,t) = 0 \; \forall (x,t) \in]0;1[\times\{0\}, k \in \{0;1\} \end{cases} \quad (1)$$

where $u(x,t)$ is the transverse displacement, α the fluid damping, β the structural damping, ϵ, the coefficient of the nonlinearity and $\phi(x)$ the spatial distribution of force $f(t)$.

– In the second model, only the boundary conditions change (NL1,B2)

$$\forall t \in \mathbb{R}^+, u(0,t) = e_0(t),$$

$$\forall t \in \mathbb{R}^+, u(1,t) = e_1(t),$$

where $e_0(t)$ and $e_1(t)$ describe a displacement or a force imposed respectively in $x = 0$ and $x = 1$. This allows to "connect" the string extremities to other mechanical system such as a bridge.

– The third model (NL2,B1) has been presented by Carrier (cf. [3]) : in our case the coupling with longitudinal displacement is neglected but damping forces are included :

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \beta \frac{\partial}{\partial t} \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\frac{\partial u}{\partial x}}{\sqrt{(1 + \frac{\partial u}{\partial x}^2)}} \right] + \phi(x).f(t) \tag{2}$$

boundary and initial conditions are the same than in the first model.

INTRODUCTION TO VOLTERRA SERIES

Volterra series are used to solve weakly nonlinear systems (in the "automaticians' meaning"). For a detailed presentation, we refer to [2, 6, 9]. This tool represents the output of a system as an infinite sum of multi-convolutions, as follows :

Definition

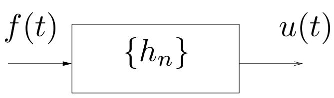

FIG. 1: System represented by a Volterra series with kernel h_n

A causal system with input f and output u is represented by Volterra series if (cf. [6])

$$u(t) = \sum_{n=1}^{\infty} \int_{(\mathbb{R}^+)^n} h_n(\tau_{1,n}) f(t - \tau_1)...f(t - \tau_n) \, \mathrm{d}\tau_{1,n} \tag{3}$$

with the notation $(\tau_{1,n}) = (\tau_1, ..., \tau_n)$ and $d\tau_{1,n} = d\tau_1...d\tau_n$. We denote, with capital letters H_n, the kernels in the Laplace domain which are defined by (cf. [6]) :

$$H_n(s_{1,n}) = \int_{(\mathbb{R}^+)^n} h_n(\tau_{1,n}) e^{-(s_1\tau_1 + ... + s_n\tau_n)} \, \mathrm{d}\tau_{1,n},$$

where $s_{1,n}$ denotes complex Laplace variables. **We will not study the convergence of the Volterra series in this paper.**

Interconnection laws

Consider two Volterra series with kernels $\{a_n\}$ and $\{b_n\}$. The systems with input f and output u defined in Fig. 2 (sum, product and cascade with a linear system) are still described by a Volterra series. Their kernels $\{c_n\}$ are given in the Laplace domain by, respectively (see [6]),

$$C_n(s_{1,n}) = A_n(s_{1,n}) + B_n(s_{1,n}) \tag{4}$$

$$C_n(s_{1,n}) = \sum_{p=1}^{n-1} A_p(s_{1,p}) B_{n-p}(s_{p+1,n}) \tag{5}$$

$$C_n(s_{1,n}) = A_n(s_{1,n}) B_1(\widehat{s_{1,n}}) \tag{6}$$

with the notation $\widehat{s_{1,n}} = s_1 + ... + s_n$.

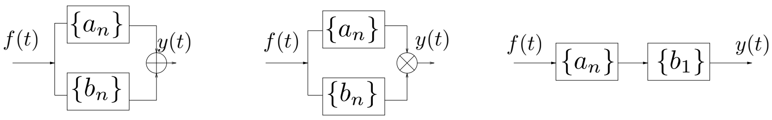

FIG. 2: Interconnection laws : Sum, product of outputs of two Volterra systems, and, cascade of a Volterra system with a linear system

SOLUTION FOR THE FIRST MODEL

In this section, the displacement $u(x,t)$ is modeled as the output of a Volterra system (the string) with input $f(t)$ (the excitation force). This involves that the Volterra kernels of this system depend on the space variable x : the solution kernels are then denoted $h_n^{(x)}$.

Resolution

Consider that the kernels $H_n^{(x)}$ are known. The first step is to built a canceling system from the partial differential equation (1), as presented through the block diagram in fig. 3.

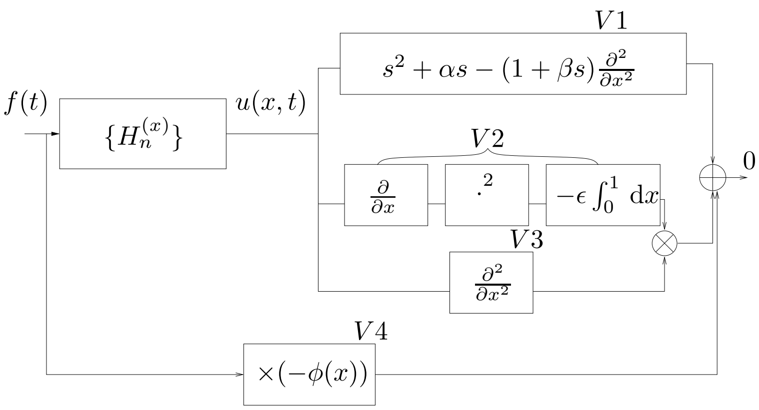

FIG. 3: Partial differential equation decomposition

Canceling system Thanks to equations (4-6), the cascade of $\{H_n\}$ with blocks V_i $(1 \leq i \leq 4)$ in fig. 3 have equivalent kernels in the Laplace domain given by, respectively :

$$
\begin{aligned}
V_1: \quad & A_n^{(x)}(s_{1,n}) &=& \left[(\widehat{s_{1,n}})^2 + \alpha(\widehat{s_{1,n}}) - (1+\beta(\widehat{s_{1,n}}))\frac{\partial^2}{\partial x^2}\right] H_n^{(x)}(s_{1,n}), \\
V_2: \quad & B_n^{(x)}(s_{1,n}) &=& -\epsilon \int_0^1 \sum_{p=1}^{n-1} \frac{\partial H_p^{(x)}(s_{1,p})}{\partial x} \frac{\partial H_{n-p}^{(x)}(s_{p+1,n})}{\partial x}\,\mathrm{d}x, \\
V_3: \quad & C_n^{(x)}(s_{1,n}) &=& \frac{\partial^2}{\partial x^2} H_n^{(x)}(s_{1,n}), \\
V_4: \quad & D_n^{(x)}(s_{1,n}) &=& \delta_{1,n}\phi(x).
\end{aligned}
$$

Then, from equation (4), $H_n^{(x)}(s_{1,n})$ satisfies the following equation

$$
(\widehat{s_{1,n}}^2 + \alpha\widehat{s_{1,n}})H_n^{(x)}(s_{1,n}) - (1+\beta\widehat{s_{1,n}})\frac{\partial^2 H_n^{(x)}(s_{1,n})}{\partial x^2} = E_n^{(x)}(s_{1,n}), \tag{7}
$$

where

$$
\begin{aligned}
E_n^{(x)}(s_{1,n}) &= D_1^{(x)}(s_1) = \phi(x)\mathbf{1}(s_1) \text{ if } n=1 \\
E_n^{(x)}(s_{1,n}) &= \sum_{p=1}^{n-1} B_p^{(x)}(s_{1,p}) C_{n-p}^{(x)}(s_{p+1,n}), \quad \forall n \geq 2
\end{aligned}
$$

Moreover the boundary conditions imply $H_n^{(x=0)}(s_{1,n}) = 0$ and $H_n^{(x=1)}(s_{1,n}) = 0$. For each n, equation 7 is linear differential equation.

Modal projection The kernels can be projected on the L^2-modal basis of the linear problem ($\epsilon = 0$). This basis is $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}^*}$ with $e_k(x) = \sqrt{2}\sin(k\pi x)$ for $x \in [0;1]$, with the scalar product $< .,. >_{L^2}$ so that

$$H_n^{(k)}(s_{1,n}) =< H_n^{(x)}(s_{1,n}), e_k(x) >= \int_0^1 H_n^{(x)}(s_{1,n}) e_k(x)\,\mathrm{d}x.$$

Equation 7 becomes (still with the notation $\widehat{s_{1,n}} = s_1 + ... + s_n$)

$$H_n^{(k)}(s_{1,n}) = \frac{E_n^{(k)}(s_{1,n})}{P_k(\widehat{s_{1,n}})}$$

with $P_k(\widehat{s_{1,n}}) = \widehat{s_{1,n}}^2 + \alpha\widehat{s_{1,n}} + (1+\beta\widehat{s_{1,n}})k^2\pi^2$ and $E_1^{(k)}(s_1) =< \phi, e_k >= \phi_k$. For $n \geq 2$,

$$E_n^{(k)}(s_{1,n}) = -\epsilon k^2\pi^2 \sum_{\substack{p,q,r\geq 1\\ p+q+r=n}} \sum_{l\in\mathbb{N}^*} \left[l^2\pi^2 H_p^{(l)}(s_{1,p}) H_q^{(l)}(s_{p+1,p+q})\right] H_r^{(k)}(s_{p+q+1,n}).$$

This yields a recurrent equation to compute kernel of order n from kernels of lower order (that are already known). Note that kernels H_n are zero for all even orders.

Simulation in the time domain

We consider the truncated series, keeping orders $n = 1,2,3$ and K modes only. This gives an approximation of the solution which is $o(\epsilon)$ for the nonlinearity.

For $n = 1$, the kernel is approximated by $\tilde{H}_1^{(x)}(s) = \sum_{k=1}^K H_1^{(k)}(s) e_k(x)$ in the Laplace domain. Each $H_n^{(k)}(s)$ corresponds to the transfer function of a standard second order filter.

For $n = 2$, all kernels $H_2^{(k)}$ ($k \in \mathbb{N}^*$) are zero.

For $n = 3$, the kernel is approximated by $\tilde{H}_3^{(x)}(s_{1,3}) = \sum_{k=1}^K H_3^{(k)}(s_{1,3}) e_k(x)$ with

$$H_3^{(k)}(s_{1,3}) = -\epsilon\, k^2\, \pi^4 \left[P_k(\widehat{s_{1,3}})\right]^{-1} \left[\sum_{l\in\mathbb{N}^*} l^2 H_1^{(l)}(s_1) H_1^{(l)}(s_2)\right] H_1^{(k)}(s_3).$$

Now, from interconnection laws, the elementary system given in fig. 4 corresponds to a Volterra system with zero kernels except that for $n = 3$ which is given by $A(s_1)B(s_2)C(s_3)D(s_1+s_2+s_3)$.

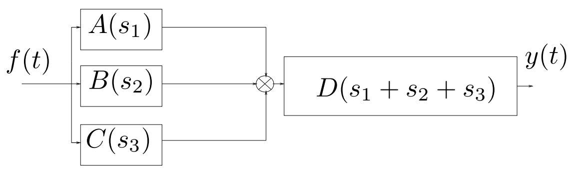

FIG. 4: Sums and products of filters to simulate Volterra kernels

Identifying $\tilde{H}_3^{(x)}(s_{1,3})$ as a sum of such elementary systems (thanks to equation (4)) leads to the structure given in fig 5. In this figure, the first column corresponds to a set of second order filters associated to the dynamics of

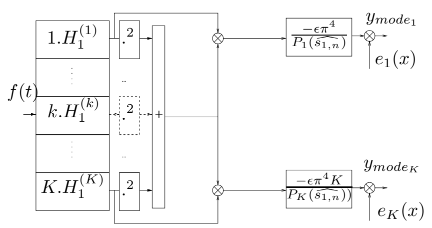

FIG. 5: Kernel over the K first modes for first model

linear modes. The second stage corresponds to instantaneous sums and products in the time domain. The last stage corresponds to a set of second order filters, the outputs of which give the third order nonlinear dynamics of each mode. The approximation of order three is obtained by adding the linear dynamics and these third order dynamics, for each mode.

SECOND MODEL

The second model has two dynamic boundary conditions, so multi-inputs have to be managed. Let's generalize Volterra series to three-inputs systems.

$$\begin{aligned} u(t) &= \sum_{\underline{m}\in\mathbb{M}^3} \int_{\mathbb{R}^{|\underline{m}|}} h_{\underline{m}}(t_{1,m_1}, \tau_{1,m_2}, \theta_{1,m_3}) f(t-t_1)...f(t-t_{m_1}) e_0(t-\tau_1)...e_0(t-\tau_{m_2}) \\ & \quad e_1(t-\theta_1)...e_1(t-\theta_{m_3})\, \mathrm{d}t_{1,m_1}\, \mathrm{d}\tau_{1,m_2}\, \mathrm{d}\theta_{1,m_3} \end{aligned}$$

with

$$\begin{aligned} \underline{m} &= (m_1, m_2, m_3) \in \mathbb{M}_3 = \{(m_1, m_2, m_3) \in \mathbb{N}^3 / m_1 + ... + m_3 \geq 1\}, \\ |\,\underline{m}\,| &= m_1 + m_2 + m_3 \end{aligned}$$

The order n of the nonlinearity become an ordered triplet (m_1, m_2, m_3) where each value is associated with an input (force $f(t)$ for m_1, displacement $e_0(t)$ for m_2 and displacement $e_1(t)$ for m_3).

Expression of the kernels The three linear kernels projected on e_k for Dirichlet-Dirichlet conditions are : $H^{(k)}_{(1,0,0)}(s) = \frac{\phi_k}{P_k(s)}$, $H^{(k)}_{(0,1,0)}(s) = \sqrt{2}\frac{(2+(-1)^{k+1}4)k\pi}{P_k(s)}$ and $H^{(k)}_{(0,0,1)}(s) = \sqrt{2}\frac{(-1)^k k\pi}{P_k(s)}$. Recurrence relation for the projected kernels for $|\,\underline{m}\,| \geq 2$ is

$$H^{(k)}_{\underline{m}} = \frac{-\epsilon k^2\pi^4}{P_k(s_{\underline{1},\underline{m}})} \sum_{\substack{\underline{p},\underline{q},\underline{r}\in\mathbb{M}_3^3 \\ \underline{p}+\underline{q}+\underline{r}=\underline{m}}} \sum_{l\in\mathbb{N}^*} l^2 H^{(l)}_{\underline{p}}(s_{\underline{1},\underline{p}}) H^{(l)}_{\underline{q}}(s_{\underline{p}+1,\underline{p}+\underline{q}}) H^{(k)}_{\underline{r}}(s_{\underline{p}+\underline{q}+1,\underline{m}}).$$

Considering this equation, it can be noted that for a given order $m_1 + m_2 + m_3 \geq 3$, each input will influence the nonlinear responses of the whole system. As in the first model, $H^{(x)}_{\underline{m}} = 0$ if $m_1 + m_2 + m_3$ is even. A structure for the simulation in the time domain, similar to that of Fig.5, can be found (see [8]).

THIRD MODEL

To define the Volterra kernels of the third model, equation (2) must be rewritten to remove fraction power (in order to use interconnection laws). To obtain a third order Volterra approximation, a Taylor series expansion is performed, up to order 3 :

$$\frac{\partial}{\partial x}\left[\frac{\frac{\partial u}{\partial x}}{\sqrt{(1+\frac{\partial u}{\partial x}^2)}}\right] \approx \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} - \frac{1}{2}(\frac{\partial u}{\partial x})^3\right].$$

Expression of kernels The linear kernel is the same as in the first model since only the nonlinearity has changed :

$$H^{(k)}_1(s_1) = \frac{\phi_k}{P_k(s_1)}$$

Following the same procedure, a recurrence relation can be found. Kernels are zero for even order and for $n \geq 3$ and $k \in \mathbb{N}^*$,

$$\begin{aligned} H^{(k)}_n(s_{1,n}) &= \frac{E^{(k)}_n(s_{1,n})}{P_k(s_{1,n})} \\ &= \frac{\pi^4}{2P_k(s_{1,n})} \sum_{i_{1,3}\in\mathbb{I}^3_n} \sum_{\underline{k}\in(\mathbb{N}^*)^3} \sum_{\underline{\xi}\in\Xi} k_1k_2k_3 H_{i_1,k_1} H_{i_2,k_2} H_{i_3,k_3} (|\,\xi^T k\,|) \int_0^1 \sin(|\,\xi^T k\,|\,\pi x)\sin(k\pi x)\,\mathrm{d}x \end{aligned}$$

with $\Xi = \{1\} \times \{-1; 1\}^2$ and

$$\mathbb{I}^p_n = \{(i_1, ..., i_p) \in (\mathbb{N}^*)^p / i_1 + ... + i_p = n\}\ p \leq n.$$

The expressions are more complex than those of previous models : they involve more terms which isolate all interactions between modes k_1, k_2 and k_3 such as $k = k_1 \pm k_2 \pm k_3$.

SOUND SYNTHESIS : SIMULATION OF THE DISPLACEMENT FOR THIRD MODEL

The sound synthesis has been realized for the three models with the same parameters. The simulation has been performed thanks to standard digital versions of second order linear filters, instantaneous sums and products. An oversampling (with factor 3) is used to remove the aliasing due to the (third order) nonlinearity.

Here, we focus on one feature of these results : the variation of the fundamental frequency for a sound computed at order 3. As shown on fig. 6, the fundamental frequency f^* is constant for the linear approximation ($n = 1$) whereas with the contribution of the nonlinearity, this frequency is lower at the attack of the sound and grows until the same value f^*, after around 500 samples for a sampling frequency $Fs = 44100$ Hz, that is, 0.01 second. This observation is similar for the three models.

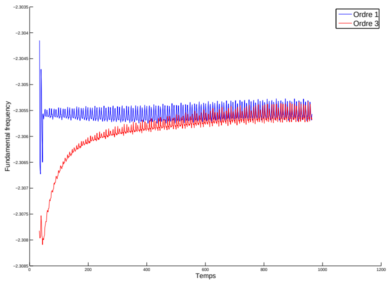

FIG. 6: Fundamental frequency variation, for the linear and the third order approximation of the third model of string. This frequency has been computed thanks to the Yin method [4].

Other results can be found in [8].

CONCLUSION AND PERSPECTIVES

This work has presented an application of Volterra series to simulate the nonlinear vibrations of a string. The results are relevant for musical acoustics. Moreover, this method allows to solve models that includes nonlinear phenomena at a given order and with a chosen number of modes. For each model, first kernels have been calculated and a recurrence relations have been found.

Studies on the convergence and the estimation of the error due to truncation have to be done. Extensions to other physical models (e.g. bi- and tri-dimensional) will be carried out, using for example, finite elements method.

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